RESISTANCE OF MATERIALS

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BY

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PREFACE

The subject of Resistance of Materials as developed in this book is divided in two parts. Part I, called Mechanics of Materials, treats chiefly of the application of the principles of Analytical Mechanics and of the experimental laws of structural materials to the analysis of the action in the members used in structures and machines. Part II treats chiefly of the structural (force-resisting) properties of engineering materials.

The main object of Part I is to develop rational methods for the design of the common types of force-resisting members used in engineering structures. The main objects of Part II are (*a*) to investigate the properties of materials from which may be determined the suitability of material for various structural uses, and (*b*) to consider tests by means of which these properties may be measured. But, attention should be called to the fact that much of the value of the study of Resistance of Materials lies in gaining correct ideas or concepts of the ways various structural and machine members act in resisting loads, and of the adjustments that occur in a member as loads are applied to it; this general knowledge of the actions in materials and members has many opportunities to function in engineering practice where definite calculations are not required and when it is impossible to obtain quantitative results.

Although Part I is self-contained, reference to Part II (as indicated throughout Part I) should prove of value to the student. Further, Part II should be of great help to the student in connection with laboratory work in materials testing.

Throughout the book consideration is given to the effects, on the analysis, of deviations in the conditions that may obtain with actual members from those that are assumed in the analysis. Further, the limitations of the methods of analysis and of the formulas developed are pointed out.

Great care has been exercised in selecting problems that are of practical value and yet are easily comprehended and are free from unimportant details so that the principles used in their solution will stand out clearly. Illustrative problems are given at the end of the more important articles and many problems are offered for solution; the answers to about one-half of the problems are given.

The moment-area principle for expressing the relations between the elastic properties of a beam and the external forces acting on the beam is treated in Chapters VIII and IX. Two methods of applying or interpreting the moment-area principle are used; namely, the slope-deviation method and the conjugate-beam method. These methods are considered mainly to be supplementary to the double-integration method treated in Chapters VI and VII. Chapters VIII and IX may be omitted without destroying the continuity of the book, and either one of the two methods of interpretation may be studied without studying the other.

Among the special features of the book may be mentioned the treatment of repeated stress and fatigue of metals (Chapter XIV); the emphasis on the principle of work and energy in determining the effect of impact loading (Chapter XIII); the use of the conjugate-beam method, in addition to the slope-deviation method, in applying the moment-area principle to beams; and the discussion, in Part II, of the significance of the structural properties of materials.

During the preparation of the manuscript the author was greatly aided by valuable suggestions and contributions from many of his colleagues. The author is especially indebted to Professor A. N. Talbot for helpful suggestions in connection with the general point of view toward the subject, and also with the content of the book and the method of treatment of various topics; to Professor H. F. Moore for valuable contributions to the content of the Chapters on repeated stress and properties of materials; to Professor N. E. Ensign for reading the manuscript and for solving most of the problems; and to Professors H. M. Westergaard and F. E. Richart for contributions to the methods of applying the moment-area principle to beams. The help thus received has contributed much to the value of the book.

F. B. S.

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RESISTANCE OF MATERIALS

PART I. MECHANICS OF MATERIALS

CHAPTER I

STRESS AND STRAIN

1. Introduction.—Resistance of Materials is that branch of Mechanics which treats of the internal forces in a physical body and of the changes of shape and size of the body, particularly in their relation to the external forces that act on the body, and to the physical properties of the material of the body. The external forces that act on the body are called *loads*; the internal forces, which resist the external forces, are called *stresses*, and the changes in the dimensions of the body are called *deformations or strains.*

The *total stress* [1] on a section through a body is the total internal force acting on the section; the component of the internal force acting normal to the area is called *normal stress*; and the component acting tangent to (or in) the area is called *shearing stress.* Further, a normal stress may be a tensile stress or a compressive stress according as the body is stretched or is shortened. *Intensity of stress* or *unit-stress* [1] is defined to be stress per unit of area. In general the intensity of stress varies from point to point over a section, its value at any point being considered to be the stress on an elementary or differential part of the area, including the point, divided by the elementary area; but when the stress is distributed uniformly on an area the intensity of stress at all points in the section is equal to the total stress on the area divided by the whole

[1] In technical literature the term "stress" is sometimes used to denote what is here defined as intensity of stress or unit-stress, and the term "internal force" is used to denote what is here called total stress.

area. A unit-stress may be expressed in various units; such as pounds per square inch (lb. per sq. in.), kilograms per square centimeter (kg. per sq. cm.), tons per square inch (ton per sq. in.), etc.

Similarly, the *total deformation* or *total strain* in any direction is the total change in the dimension of the body in that direction, and the *unit-deformation* or *unit-strain* in any direction is the deformation or strain per unit of length in that direction.

The Problem Defined.—A body when subjected to loads is stressed and deformed, and the values of the stresses and the deformations in the body are of great importance in many engineering problems. These stresses are found by use of the general principles of mechanics (mainly of statics) and of the experimental laws that have been found to govern the action of the material. The main objects, then, in the study of Resistance of Materials are

> (1) To determine the relation between the external forces (loads) acting on a body and the resulting internal forces (stresses) in the material so that stresses may be determined from known loads, or so that the loads that produce give stresses may be found, and
>
> (2) To determine the relation between loads acting on a body and the resulting strains produced in the body so that strains may be determined from known loads, or vice versa, and
>
> (3) To obtain a knowledge of the physical properties, such as, stiffness, strength, ductility, toughness, resilience, etc., of the various structural materials, since physical properties of the materials are involved in the relations under (1) and (2) and are of special importance in determining the suitability of a material for resisting loads under varying conditions of loading.

That part of Resistance of Materials which considers chiefly the application of the principles of mechanics to materials for the purposes stated under (1) and (2) above is frequently called "Mechanics of Materials" and is discussed mainly in Part I of this book, whereas Part II is devoted to a brief discussion of the properties of structural materials.

As suggested by the statements under (1) and (2) above, in some problems of design, the stress produced in the body by the load is the governing factor in the design, whereas in other problems the strain produced is the more important factor. For example, the chain of a hoist fulfills its function of lifting loads regardless of the amount of stretch of the chain that occurs, provided that the load does not produce too large a stress in the chain. That is, the stress developed in the chain by the maximum load to be applied is the governing factor in the design of the chain. Likewise, stress is generally the governing factor in the design of many of the parts of bridges, buildings, cranes, ships, etc. In the design of machine tools, however, such as planers, lathes, drill presses, grinding machines, etc., the deformations of the parts are frequently of prime importance since such machines will not produce work of sufficient accuracy if the deformation of the parts is too large.

In obtaining the relations under (1) and (2) above, it will be found that the stresses and strains produced in a body by the loads depend on (*a*) the type of loading (whether static loads, impact loads, or repeated loads), (*b*) the dimensions of the body or the shape of area on which the stress occurs, and (*c*) the properties of the material of the body. An example of the influence of each of the above factors may be given as follows: (*a*) a load applied suddenly to a body produces more stress and deformation in the body than does the same load when applied slowly; (*b*) a rolled I-section or channel-section has its area distributed better for resisting bending than does a bar of equal area having a rectangular or circular cross-section; (*c*) the elongation of an oak stick caused by a given load is greater than that of a bar of steel of the same dimensions when subjected to the same pull, etc., etc.

2. Types of Loading.—With reference to the manner in which loads are applied or transmitted to a structure or machine, the loads will be considered under three distinct headings; namely, static loads, impact and energy loads, and repeated loads.

1. *Static*, *steady* or *dead* loads are forces that are applied slowly and not repeated, and remain nearly constant after being applied to the body, or are repeated relatively few times; such as the loads on most buildings (a part of the load being the weights of the members of the structure), the load applied to a bar in a testing machine, etc.

2. *Impact* loads are forces that are applied to the resisting body in a relatively short period of time; the shorter the time the greater the effect of the impact. An impact load, in general, is applied by a body that is in motion when it comes in contact with the resisting body, and the force exerted by the moving body and the period during which it acts can not in general be determined. For this reason in many problems it is more satisfactory to calculate the stress and strain produced by an impact load from the energy delivered to the resisting body by the moving body. When this is done the energy delivered to the resisting body is called an *energy load* and is expressed in foot-pounds (not in pounds). Impact and energy loads are considered in Chapter XIII.

3. *Repeated* loads are forces that are applied a very large number of times causing a stress (or stresses) in the material that is continually changing, usually through some definite range. For example, the loads applied to the connecting rod of an engine when the engine is running, the wheel loads on a railroad rail as a train passes over the rail, etc., are repeated loads. Repeated loads are discussed in Chapter XIV.

Other Classifications of Loads.—Loads may be classified as *distributed* loads and *concentrated* loads. A distributed load may be uniformly distributed or non-uniformly distributed. Thus, if sand be spread on a floor so that its depth is constant, the floor will be subjected to a uniformly distributed load, whereas, if the sand be distributed so that its depth is not constant the floor is said to carry a non-uniformly distributed load. A concentrated load is one whose area of contact with the resisting body is negligible in comparison with the area of the resisting body.

With reference to the manner in which the stresses in a body will vary from point to point and also to the general kind or type of stress and deformation that will be developed, the loads will, for convenience, be considered under three headings; namely, *central* loads, *torsional* loads, and *bending* loads. A body, however, may be subjected simultaneously to loads of any two or to all three of these types.

The stresses and strains caused by static central loads are discussed in this chapter. The stresses and strains caused by static torsional loads are considered in Chapter IV, and those produced by static bending loads in Chapter V.

Static Central Loads

3. Stresses Due to Central Loads.—A central load is a concentrated load whose action line passes through the centroid of the area on which the stresses are to be considered, or a distributed load whose resultant passes through the centroid of the area. The stress produced by a central load may be any one of the three types, tensile stress, compressive stress, or shearing stress, but the distinguishing feature of a central load as compared with torsional and bending loads is that the stress may be assumed to be uniformly distributed over the area, that is, the intensity of stress (unit-stress) may be assumed to be constant. If a central load acts normal to the area it is called an *axial* load and it may be a

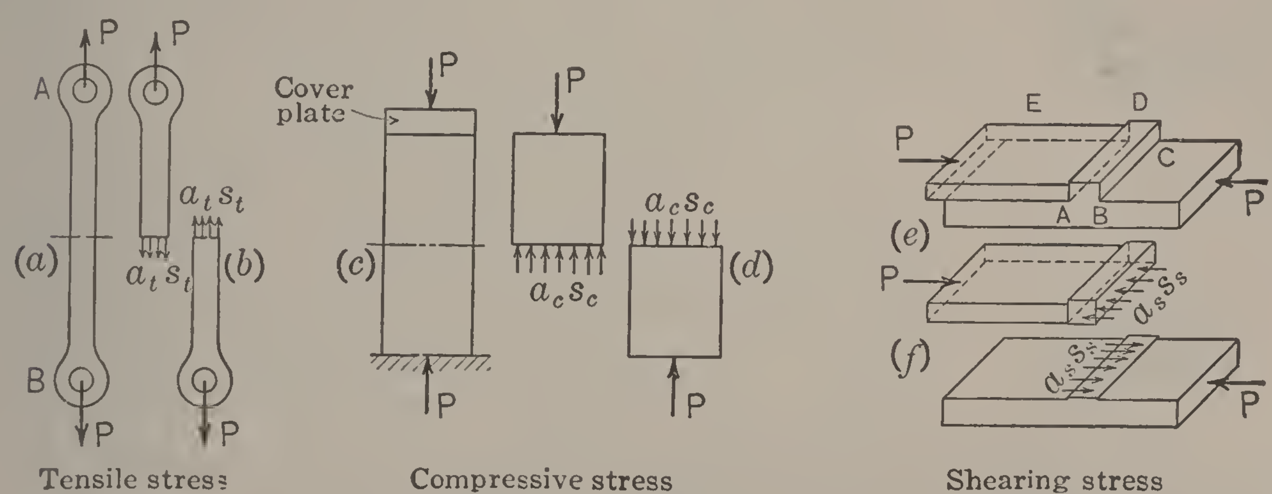


Fig. 1.—Tensile, compressive and shearing stresses.

tensile or a compressive axial load according as it produces tensile or compressive stress; if the central load lies in the plane of the area it is called a shearing central load.

Stresses due to central loads may be found as follows: Fig. 1(*a*) represents a straight bar AB subjected to an axial tensile load P causing tensile stress on any cross-section of the bar; Fig. 1(*c*) represented a bar in compression under the action of an axial load P causing compressive stress in the bar; and Fig. 1(*e*) represents a body under the action of a central shearing load P causing shearing stress on the area $ABCD$. The problem is to find the relation between the load and the unit-stress developed for each of these three types of central loads. Let a section be passed through the body in each case and a free-body diagram for either part of the body be drawn, as shown in Figs. 1(*b*), 1(*d*), and 1(*f*). Since the load in each case passes through the centroid of the area, the intensity of stress may be assumed to be constant; that is, the stress may be assumed to be uniformly [2] distributed over the area, and hence the total stress (called the resisting stress) is equal to the product of the area, a, and the unit-stress, s. Therefore, since each part of the body, in each case, is in equilibrium under the action of two collinear forces we may write

load = resisting stress.

Hence,

$$P = a_t s_t; \quad P = a_c s_c; \quad \text{and} \quad P = a_s s_s, \quad . \quad . \quad . \quad . \quad (1)$$

in which the subscript denotes the kind of stress (tensile, compressive and shearing). Thus, in general, a central load, for any section of a body, is equal to the product of the area of that section and the unit-stress on the area.

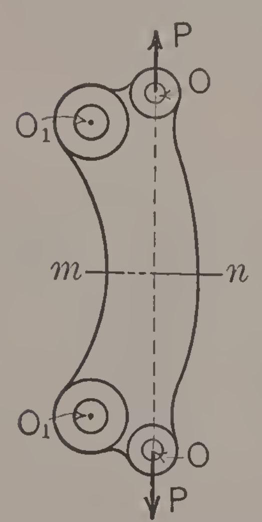


FIG. 2. — Central load with respect to one section only.

It should be noted that the load P of Fig. 1(e) is not a central load unless the plate E is exceedingly thin, but in engineering computations shearing loads may frequently be assumed, without serious error, to be central loads even though they do not comply strictly with the above definition of a central load. Further, the loads P of Fig. 2 are axial loads for section mn but not for any other section, and if the loads were applied at points O_1 they would not be axial loads for any section of the body.

ILLUSTRATIVE PROBLEM

Problem 1. In determining the strength of concrete, a test cylinder 8 in. in diameter and 16 in. high is loaded in a testing-machine as shown in Fig. 3(a). (a) If the maximum axial load Q that the concrete specimen can resist is 100,000 lb., determine the maximum unit-stress developed in the concrete. The dimensions shown in Fig. 3(b) and 3(c) were obtained from a testing-machine having a capacity of 100,000 lb.; when the concrete specimen is resisting its maximum load find (b) the maximum tensile unit-stress in each of the two screws of the testing-machine, (c) the shearing unit-stress in

[2] For a discussion of the influence of the relative dimensions of the body and of other conditions on the distribution of the stress over the cross-section, see Art. 136 to 139.

the threads of the screws, (*d*) the shearing unit-stress on the cylindrical area AB of the bronze bushing, and (*e*) the bearing unit-stress of the threads on the bushing.

Solution.—(*a*) Since the load Q on the concrete specimen is an axial load the unit-stress on any section mn is uniformly distributed, and hence the total resisting stress is equal to $a_c s_c$ (see Fig. 1*d*). Hence,

$$Q = a_c s_c,$$

$$100{,}000 = \frac{\pi(8)^2}{4} \cdot s_c,$$

$$s_c = 1990 \text{ lb. per sq. in.}$$

Thus, the compressive stress in the concrete is 1990 lb. per sq. in.

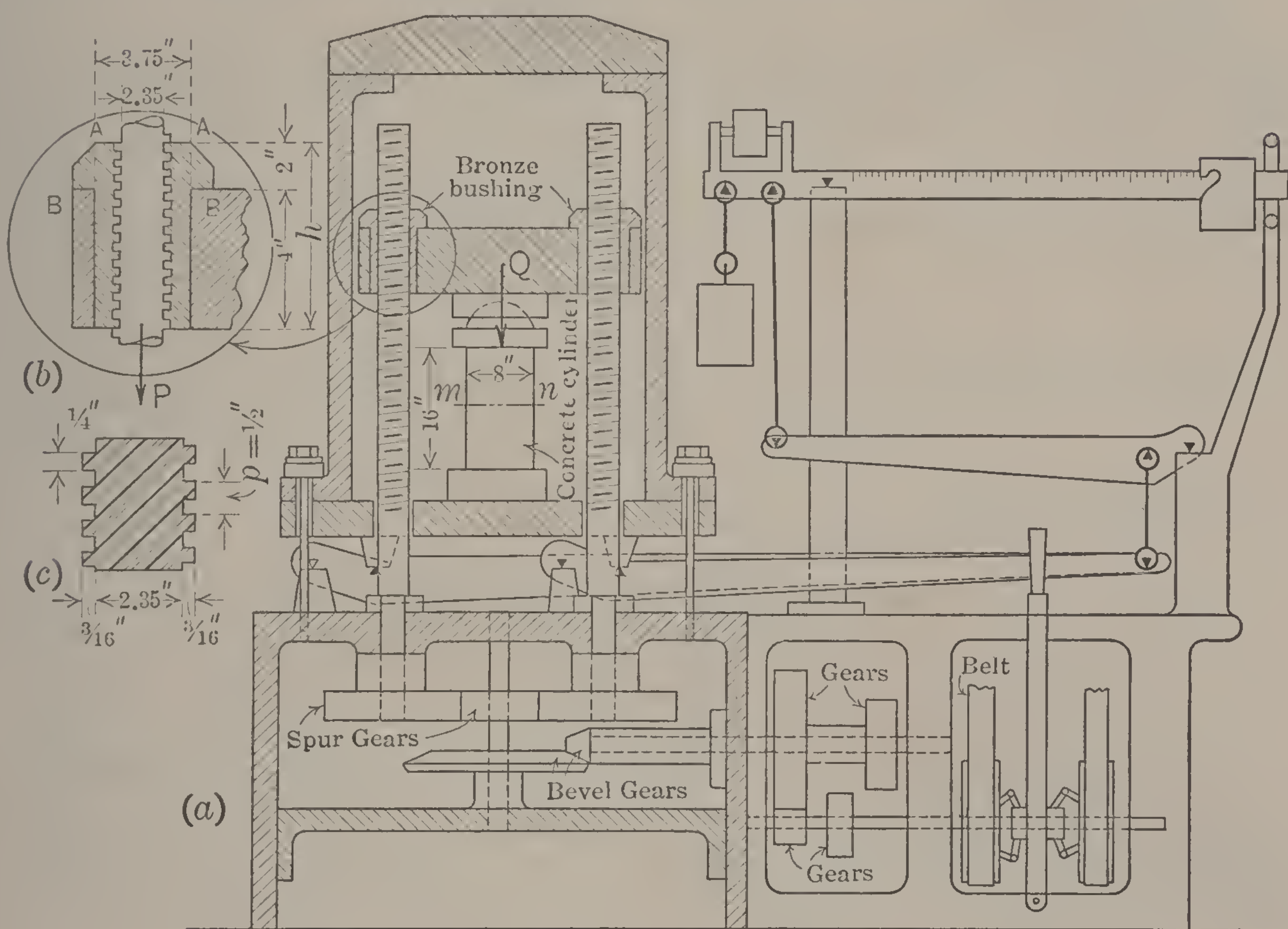


FIG. 3.—Testing machine.

(*b*) A tensile load P of 50,000 lb. is resisted by each screw. Hence,

$$P = a_t s_t,$$

$$50{,}000 = \frac{\pi(2.35)^2}{4} \cdot s_t,$$

$$s_t = 11{,}500 \text{ lb. per sq. in.}$$

The maximum tensile stress, then, that will be developed in the screws of this machine is 11,500 lb. per sq. in.

(*c*) The area of the thread on which the shearing stress is developed as the thread resists being stripped from the screw is, Shearing area $= a_s =$ thickness of thread $\times$ circumference at root of thread $\times$ number of threads, n, in the depth, h, of the bushing in which the screw turns $\left(n = \frac{h}{p},\right.$ where p is the pitch)

$$= \frac{1}{4} \times \pi \times 2.35 \times \frac{6}{0.5} = 22.15 \text{ sq. in.}$$

Now since the load of 50,000 lb. on each screw is resisted by the shearing stress on this 22.15 sq. in. of shearing area, we have,

$$P = a_s s_s,$$

$$s_s = \frac{50{,}000}{22.15} = 2260 \text{ lb. per sq. in.}$$

(*d*) The shearing area in the head of the bushing is

$$a_s = \text{circumference} \times \text{depth}$$
$$= \pi\, 3.75 \times 2 = 23.55 \text{ sq. in.}$$

The shearing unit-stress on this area due to the 50,000-lb. load is

$$s_s = \frac{P}{a_s} = \frac{50{,}000}{23.55} = 2120 \text{ lb. per sq. in.}$$

(*e*) The bearing area, a_b, of the threads on the bushing is

$$a_b = \tfrac{3}{16} \times \pi\, 2.35 \times 12 = 16.6 \text{ sq. in.}$$

The bearing unit-stress, then, is

$$s_b = \frac{P}{a_b} = \frac{50{,}000}{16.6} = 3010 \text{ lb. per sq. in.}$$

PROBLEMS

2. If a specimen of wood is tested as shown in Fig. 4, and the maximum tensile unit-stress that the specimen can resist is 8000 lb. per sq. in., (*a*) what is the maximum axial load P that can be applied to the specimen? (*b*) What is the shearing unit-stress in the heads of specimen when the load P is applied?

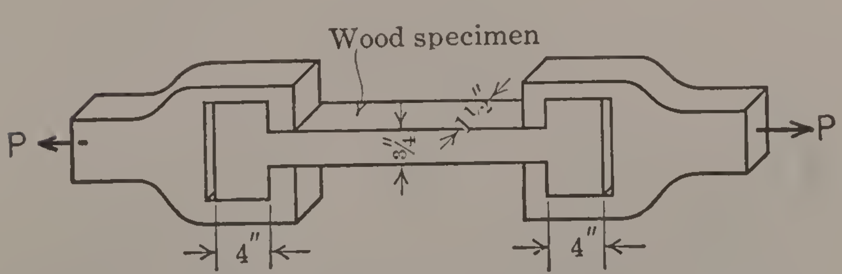


FIG. 4.—Tension and shear in wood specimen.

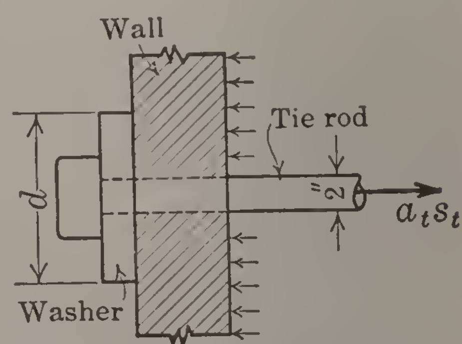


FIG. 5.—Bearing pressure of washer on wall.

3. A tie-rod 2 in. in diameter (Fig. 5) is used to help resist the lateral pressure against the walls of a bin. If the tensile unit-stress in the tie-rod is 15,000 lb. per sq. in., what diameter, d, should the washer have in order to keep the bearing unit-stress of the washer on the wall from exceeding 200 lb. per sq. in.? *Ans.* 17.5 in.

4. A timber frame shown in Fig. 6 carries a load, Q, of 12,000 lb. Find (a) the compressive unit-stress in the members A and B; (b) the shearing unit-stress in the timber C; and (c) the compressive (or bearing) unit-stress of C on the blocks D, which are 4-in. cubes.

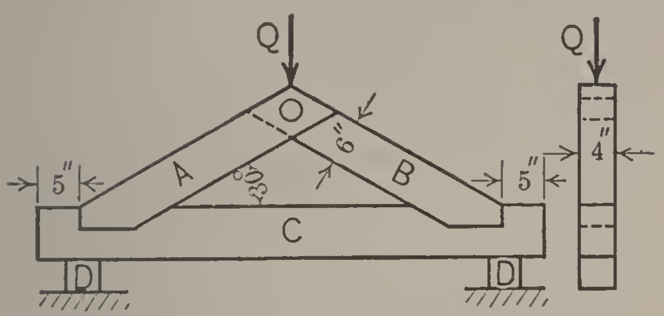


FIG. 6.—Timber frame.

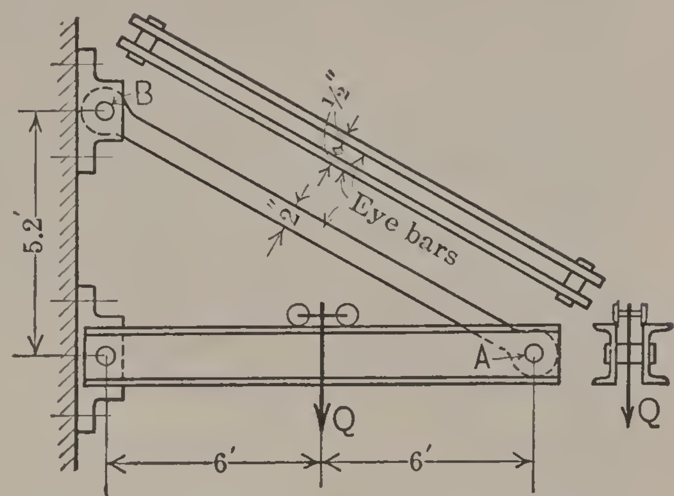


FIG. 7.—Pin-connected wall bracket.

5. The wall bracket shown in Fig. 7 carries a load, Q, of 10 tons. Find the tensile unit-stress in each of the two eye-bars, and the shearing unit-stress in the pins A and B if the diameter of each pin is $1\frac{1}{2}$ in.

Ans. $s_t = 12{,}900$ lb. per sq. in. $s_s = 7310$ lb. per sq. in.

6. The bearing unit-stress for the collar bearing shown in Fig. 8 is 90 lb. per sq. in., and the compressive unit-stress in the shaft is 6000 lb. per sq. in. If the diameter, d_1, of the shaft is 6 in., what is the load, P, on the shaft and the diameter, d_2, of the collar? If the thickness, t, of the collar is $1\frac{1}{2}$ in., what is the shearing unit-stress on the area of contact between the collar and shaft?

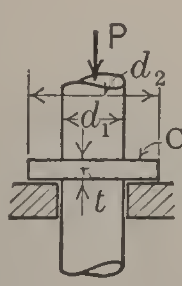


FIG. 8.—Collar bearing.

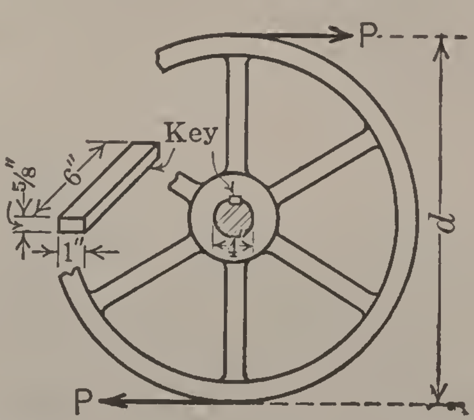


FIG. 9.—Shear in key connecting pulley and shaft.

7. A pulley (Fig. 9) transmits a turning moment, $P \times d$, of 12,000 lb.-ft. to a 4-in. shaft, relative motion between the pulley and the shaft being prevented by a flat key 1 in. wide, $\frac{5}{8}$ in. deep, and 6 in. long. Compute the shearing unit-stress in the key. *Ans.* $s_s = 12{,}000$ lb. per sq. in.

4. Strains Due to Central Loads.—*Tensile Strain.*—If an axial tensile load is applied to a straight bar of constant cross-section and of homogeneous material the bar is elongated or stretched and the strain per unit of length, that is, the unit-strain (unit elongation) ϵ_t is given by the expression

$$\epsilon_t = \frac{e_t}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which e_t is the total tensile deformation and l is the original length of the bar. If the cross-section of the bar is not constant or if the material is not homogeneous, all unit lengths of the bar will not elongate the same amount, and the above expression then represents only the average unit-strain; thus, since the unit-strain varies from section to section along the bar, the unit-strain at any section of the bar is the ratio of the elongation de_t, of an elemental length dl, including the section, to the length dl. That is, the unit-strain at any section of the bar is

$$\epsilon_t = \frac{de_t}{dl}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In order to determine ϵ_t from the above expression e_t of course, must be expressed in terms of l (see the solution of Problem 12 where this is done).

Compressive Strain.—Similarly, if an axial compressive load is applied to a straight bar of constant cross-section and of uniform material (see Fig. 1*c*) the bar is deformed (shortened) an amount e_c and the unit-strain (unit-shortening) ϵ_c is

$$\epsilon_c = \frac{e_c}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In loading a physical bar, it is practically impossible to obtain an axial compressive load; further, physical bodies are never homogeneous nor straight. Therefore, a compression member if relatively slender will bend when subjected to a load that is assumed to be axial. Bending action in compression members is discussed in Chapter XI.

Shearing Strain.—Shearing strain (sometimes called detrusion) usually occurs in combination with tensile and compressive strain in connection with twisting and bending action, and it usually varies from point to point in the body. Although shearing strain due to a central shearing load cannot be found experimentally as can tensile and compressive deformations, the meaning of shearing strain and the quantitative measure of shearing unit-strain will be discussed at this point. Thus, let Fig. 10(a) represent a bolt subjected to the shearing load P which tends to shear off the head of the bolt. The rectangle $ABCD$ of Fig. 10(b) represents a vertical section (enlarged) of that portion, $ABCD$, of the bolt that is subjected to shear. The head of the bolt exerts shearing stresses on the face BC and the lower part of the bolt exerts opposite shearing stresses on the face AD. Other forces would have to act on $ABCD$ to hold it in equilibrium (that is P is not, strictly speaking, a *central* shearing load for section AD), but only the shearing forces will here be considered. The shearing forces shown in Fig. 10(b) cause the rectangle $ABCD$ to assume the from of a rhombus AB_1C_1D. The part $ABCD$ of the bolt subjected to shear may be considered to be made up of thin layers of materials each layer sliding a small amount relative to the layer beneath it. The total sliding, that is, the total shearing strain in the length AB or l is e_s, and the shearing deformation per unit length, that is, the shearing unit-strain ϵ_s, is

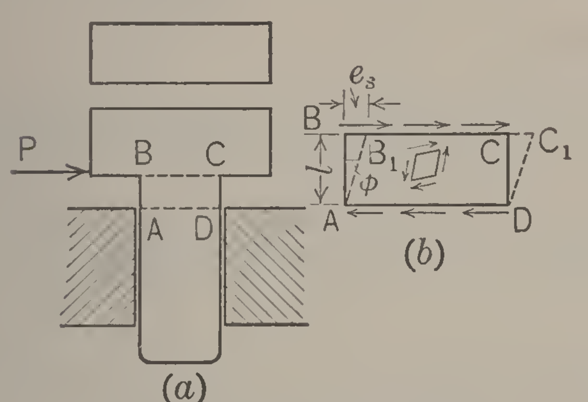


Fig. 10.—Shearing strain.

$$\epsilon_s = \frac{e_s}{l},$$

but

$$\frac{e_s}{l} = \tan\,\phi,$$

where ϕ is the change in the inclination of a line in the body that was originally perpendicular to the direction of the shearing deformation. Further, for a small angle, the tangent of the angle and the angle (expressed in radians) may be assumed to be equal without introducing serious errors. Therefore,

The shearing unit-strain at any point in a body is measured by the change in inclination (expressed in radians) of two lines that pass through the point and that were originally at right angles.

Hence

$$\epsilon_s = \frac{e_s}{l} = \tan \phi = \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

It will be shown later that a shearing stress (and also strain) on a given plane at a point in a body requires that there be a shearing stress (and strain) of equal intensity at the same point on a plane at right angles to the first plane, as is suggested by the distortion of the small cube within the body $ABCD$ as shown in Fig. 10(b).

It will be noted that a unit-strain (whether tensile, compressive, or shearing) is a length divided by a length and hence is an abstract number when each length is expressed in the same units; that is, its value is the same whether expressed as inches per inch or as feet per foot, etc.

PROBLEMS

8. A straight bar 6 ft. long and $\frac{7}{8}$ in. in diameter is turned down to a diameter of $\frac{1}{2}$ in. for a distance of 2 ft. in its central portion. An axial load P causes a unit-deformation of 0.001 in the central 2 ft. and a total stretch of 0.04 in. in the whole bar. What is the unit-deformation of each of the end portions?

9. From compressive tests of concrete cylinders it has been found that the concrete fails when the unit-deformation is about 0.0012. How much does a specimen 8 in. in diameter by 16 in. high shorten before failure occurs?

5. Stress-strain Curve. Proportional Limit. Yield-Point. Ultimate Strength. Elastic Limit. Modulus of Elasticity.— Experiments have shown that for nearly all structural materials the unit-stress in a material is approximately proportional to the accompanying unit-strain of the material provided that the unit-stress does not exceed a certain value. For example, let a steel bar (Fig. 11) of length l and of constant cross-section, a, be subjected to an axial tensile load that gradually increases from zero value until the bar breaks. If P denotes any value of the load, then the unit-stress, s_t, corresponding to that load will be $\frac{P}{a}$. Further, let measurements of the stretch, e, of the bar be taken (by means of an extensometer) for various values of s_t; the unit-strain ϵ, then, may be found from the expression $\epsilon = \frac{e}{l}$. The relation between the unit-stress, s_t, and the unit-strain ϵ, found experimentally as indicated above, is represented, within quite approximate limits, by the *stress-strain* graph in Fig. 12(a), if the material is ductile such as low-carbon steel and other ductile metals, and by the stress-strain graph in Fig. 13 if the material is brittle,[3] such as high-carbon steel, etc.

FIG. 11.—Tensile test specimen.

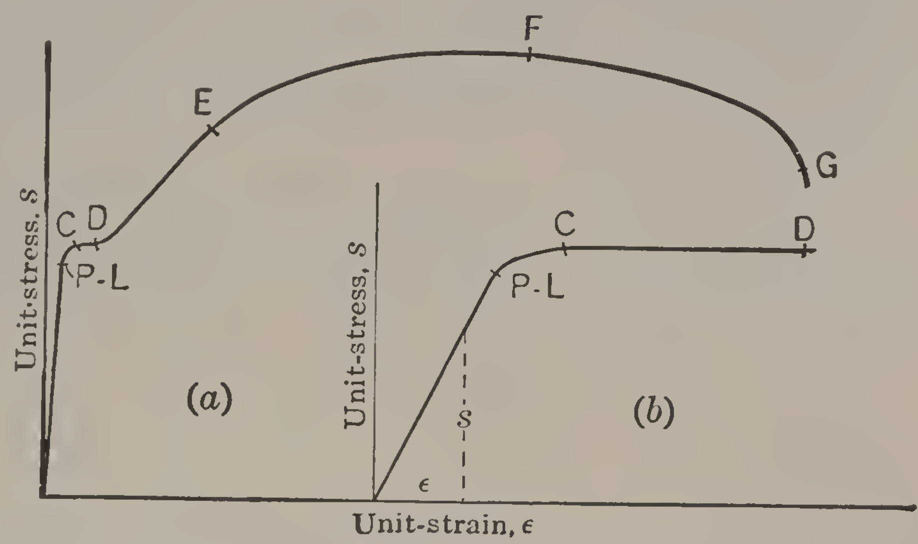


FIG. 12.—Stress-strain diagram for ductile steel.

Proportional Limit.—As indicated in Figs. 12 and 13, it is found that for most structural materials, as the unit-stress is increased the unit-strain is increased in practically the same ratio; if the unit-stress is doubled the unit-strain is likewise doubled, etc.; that is, the stress-strain curve is a close approximation to a straight line[3] until the unit-stress reaches a value called the proportional limit (or limit of proportionality). This unit-stress is represented on the stress-strain curve by the ordinate to the point P–L in Figs. 12 and 13. Therefore, the *proportional limit* of a material is defined to be the maximum unit-stress that can be developed in the material without causing the unit-strain to increase at a greater rate than the unit-stress increases, as load is applied to the body.

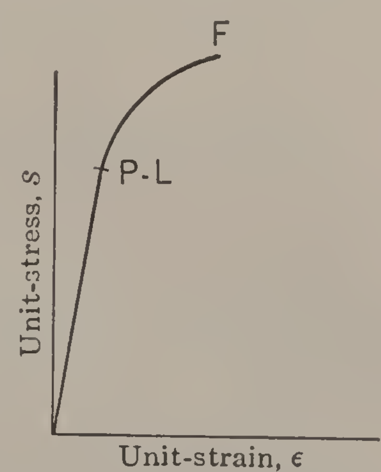


FIG. 13.— Stress-strain diagram for brittle material.

Yield Point.—As the load on the bar is increased further, causing a stress greater than the proportional limit, a unit-stress

[3] The stress-strain curves for concrete and cast iron are curved practically all the way, but the first part of the curves, corresponding to relatively small stresses, may be assumed without serious error to be straight lines.

is reached at which the material continues to deform without an increase in load, provided that *the material* is *ductile.* The unit-stress at which this action occurs is called the yield-point, and is represented on the stress-strain curve by the ordinate to the part CD of the graph in Fig. 12 (only ductile metals have yield-points). Thus the yield-point of a material is defined to be the unit-stress in the material at which the material deforms appreciably without an increase of load.

The value of the unit-deformation for a bar when stressed to its proportional limit depends on the kind of material; the value for structural steel is about 0.0012, and for timber about 0.0020. Thus, if the bar in Fig. 11 were structural steel and its length, l, were 10 in., the total stretch e when the bar is stressed to its proportional limit would be $e=\epsilon l=0.0012\times 10=0.012$ in. Thus, in order to detect and measure the small strains that accompany stresses less than the proportional limit a strain-measuring apparatus (extensometer) is attached to the specimen. Further, the load is applied and measured by means of a testing machine similar to that shown in Fig. 3. The stretch that occurs while the bar is stressed at its yield-point, however, is relatively large, being, in the case of structural steel, as much as 0.025 in. per inch of length; a bar 10 in. long, then, would stretch one-fourth of an inch without an increase in load; this stretch is represented by the portion CD of the curve in Fig. 12.

Ultimate Strength.—If the load on the bar (Fig. 11) increases still further, the unit-stress and unit-deformation increase as indicated by the portion of the curve DEF (Fig. 12a) if the material is ductile, until the maximum unit-stress is reached, which is represented by the ordinate to the curve at F and is called the ultimate strength. The ultimate strength for a brittle material is represented by the ordinate to F in Fig. 13; a brittle material breaks when stressed to the ultimate strength whereas a ductile material continues to stretch. Hence, the *ultimate strength* of a material is defined to be the maximum unit-stress that can be developed in the material, as determined from the original cross-section of the bar or specimen; the cross-section of the bar decreases somewhat as the bar is stressed above the yield-point.

After the ultimate strength of a ductile material is developed, the bar begins to " neck down," thereby rapidly reducing the area of cross-section at the neck-down section (Fig. 14), and the load required to cause the bar to continue to stretch decreases, as indicated by the curve FG (Fig. 12a). The load on the bar at the instant of rupture is called the breaking load. The breaking load divided by the area of the neck-down section is the value of the unit-stress in the bar when rupture occurs and this value is considerably greater than the ultimate strength. The ultimate strength, however, is of more importance than any stress in the bar after necking down has started, since the bar is in the process of failing after necking down starts.

Elastic Limit.—If the load on the bar is released after the bar has been stressed beyond the yield-point, the bar will not

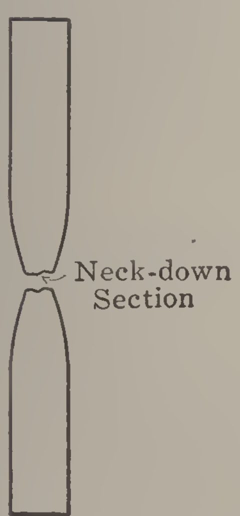


FIG. 14.—Form of ruptured specimen of ductile steel.

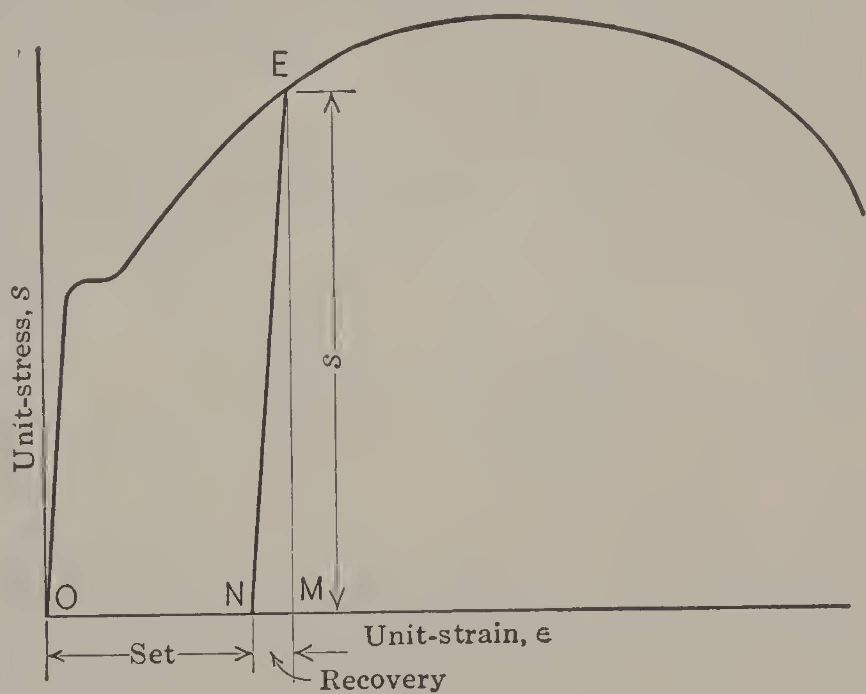


FIG. 15.—Stress-strain diagram showing permanent set.

return to its original length, but will retain a part of its deformation. The deformation per unit of length retained by the bar after the load (and stress) has been reduced to zero is called the *permanent set* or merely *set*. For example, let the bar (Fig. 11) be stressed to the unit-stress, s, represented by the ordinate, $\overline{ME}$, to the point E on the stress-strain curve in Fig. 15, the unit-strain corresponding to the unit-stress s being $\overline{OM}$. If the load is gradually released the stress-strain curve will be represented by the line EN. That is, part of $\overline{OM}$, represented by $\overline{MN}$, is recovered but a part, represented by $\overline{ON}$, is retained by the bar. $\overline{ON}$ therefore, represents the *set* corresponding to the unit-stress s.

If however, the bar were subjected to a unit-stress equal to (or less than) the unit-stress called the elastic limit, the bar will return to its original length when the load (and stress) is reduced to zero. Hence, the *elastic limit* of a material is defined to be the maximum unit-stress that can be developed in the material without causing a permanent set.

In order to determine experimentally the value of the elastic limit for a material, a bar is subjected to a relatively small axial load and the load is released; if the extensometer shows that the bar has acquired no permanent deformation a larger load is applied and then released, etc. Thus, the stress in the bar is increased in small increments until a unit-stress is found which, on release of the stress, leaves a very small set in the bar, the minimum unit-stress at which set first occurs being the elastic limit. The results of tests show that for most structural metals the elastic limit of the metal has approximately the same numerical value as has the proportional limit of the material, and in technical literature, the proportional limit frequently is called (though incorrectly) the elastic limit. Further, the yield-point is sometimes called the commercial elastic limit.

Modulus of Elasticity.—If the bar discussed in the preceding article were subjected to a compressive stress, or to a shearing stress instead of a tensile stress, the stress-strain curve would be of the same form as that shown in Figs. 12 and 13. Hence, it follows that when a material is stressed *in one direction only*, the unit-stress at any point in a material is proportional to the unit-strain at that point, provided that the unit-stress does not exceed the proportional limit of the material. That is, for stresses within the proportional limit, all elastic material behaves according to the same law; namely, that the ratio of the unit-stress to the unit-deformation is a constant, or expressed mathematically,

$$\frac{s}{\epsilon} = \text{a constant}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

regardless of the kind of stress developed (whether tensile, compresive or shearing) and regardless of the way the stress and deformation are produced (whether by axial, bending or torsional loads, etc.). This is known as *Hooke's law*.[4] The numerical value of

[4] As pointed out earlier in this article, this law should be regarded as approximate, the error involved being negligible for most computations involving static loads and ductile materials but not necessarily negligible when repeated loads are involved, as will be discussed in Chapter XIV.

this constant ratio, however, is in general different for any one material when subjected to the different types of stress, and is also different for the different materials subjected to the same types of stress.

The numerical value of the constant ratio of the unit-stress in a material to the accompanying unit-strain within the proportional limit is called the *modulus of elasticity of the material.* The symbol E will be used to denote the modulus of elasticity, and subscripts t, c and s will be used to denote tensile, compressive and shearing, respectively. Thus, we may write

$$E_t = \frac{s_t}{\epsilon_t}, \quad E_c = \frac{s_c}{\epsilon_c} \quad \text{and} \quad E_s = \frac{s_s}{\epsilon_s} \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

and, *when the stress and deformation are caused by central loads,* the above expressions may be written as follows:

$$E_t = \frac{\frac{P}{a}}{\frac{e_t}{l}} = \frac{Pl}{ae_t}, \quad E_c = \frac{Pl}{ae_c} \quad \text{and} \quad E_s = \frac{\frac{P}{a}}{\frac{e_s}{l}} = \frac{P}{a\phi} \quad . \quad . \quad . \qquad (8)$$

The value of E_s is not found experimentally, however, from the above expression since the shearing deformation ϕ (see Art. 4) caused by a central shearing load is very difficult if not impossible to measure. The value of E_s, however, can easily be found when the shearing unit-stress and shearing unit-strain are produced by torsional loads as is discussed in Chapter IV.

It should be noted that the modulus of elasticity is expressed in the same units as is unit-stress since ϵ is a ratio of length to length and is therefore merely a number.

As stated above, the values of the moduli of elasticity for any one material are, in general, not the same. For steel the tensile and compressive moduli of elasticity, E_t and E_c, are approximately 30,000,000 lb. per sq. in., but the shearing modulus, E_s, is approximately 12,000,000 lb. per sq. in. Similarly, any one of the moduli has different values for different materials. Thus, average values of the tensile moduli for steel, cast iron and timber (pine) are respectively 30,000,000, 15,000,000 and 1,500,000 lb. per sq. in.

It follows from the definition that the value of the modulus of elasticity of a material is represented by the slope of that portion of the stress-strain curve below the proportional limit; this portion of the curve is usually drawn to a large scale as in Fig. 12*b* so that the value of the slope can be obtained with reasonable accuracy (see also Art. 144). Now this slope represents the rate at which unit-stress increases with unit-strain and hence the modulus of elasticity of a material is a measure of the stiffness of the material. That is, if one material has a modulus of elasticity twice as great as another material, the resisting stress in the one material, for a given strain, is twice as great as that in the other, and hence the one material is twice as stiff as the other.

6. Properties of Structural Materials.—The proportional limit, elastic limit, yield-point, ultimate strength, modulus of elasticity, etc., of a material are usually called physical (or mechanical) properties of the material, the values of which are found from experimental results, and the numerical values of the properties are frequently called physical constants. Much of the material used in structures and machines is bought according to specifications which give the values of the physical properties that the material shall have for various uses. A number of technical societies and engineering companies have published specifications of structural materials; perhaps the most complete specifications are those of the American Society for Testing Materials called the " A.S.T.M. Standards."

Average values of various properties for a few of the more common engineering materials are given in Table 1. Attention is called to the fact that the properties of a commercial material necessarily vary somewhat due to the uncontrollable factors in the method of manufacture, treatment, etc., and that the specifications usually make reasonable allowance for this variation by specifying a range of values or a minimum value for the properties which a material must have. For example, the ultimate strength of " Structural Steel for Building " as given in the " A.S.T.M. Standards " is 55,000 to 65,000 lb. per sq. in. and the yield-point must be not less than one-half of the ultimate strength.

The average values in Table 1 are presented with the view of helping the student to develop judgment in the use of engineering materials; the values in the table should be used when needed in the solution of subsequent problems. A detailed dis-

TABLE 1

Average Values of Strength, Stiffness, and Ductility of Various Structural Materials

Material	Tensile Strength (lb. per sq. in.)		Compressive Strength (lb. per sq. in.)		Shearing Strength (lb. per sq. in.)		Modulus of Elasticity (lb. per sq. in.)		Elongation in 2 In. (per cent)
	Proportional Limit	Ultimate Strength	Proportional Limit	Ultimate Strength	Proportional Limit	Ultimate Strength	Tensile and Compressive	Shearing	
Wrought iron	30,000	60,000	30,000	(*a*)	18,000	40,000	25,000,000	10,000,000	35
Structural steel (about 0.30% carbon)	35,000	65,000	35,000	(*a*)	21,000	45,000	30,000,000	12,000,000	30
Hard (high carbon) steel (about 0.60% carbon)	60,000	110,000	60,000	(*a*)	36,000	85,000	30,000,000	12,000,000	15
Nickel steel (3.5% nickel)	50,000	90,000	50,000	(*a*)	30,000	65,000	30,000,000	12,000,000	25
Chrome-nickel steel, heat treated {0.40% carbon, 1.5 % nickel, 1.2 % chrome}	115,000	145,000	115,000	(*a*)	70,000	105,000	30,000,000	12,000,000	15
Gray cast iron	(*b*)	20,000	25,000	75,000	(*c*)	(*c*)	15,000,000	6,000,000	Negligible
Bronze, rolled (copper 95%, tin, 5%)	40,000	65,000	....	....	....	......	14,000,000	6,000,000	30
Brass, rolled (copper 60%, zinc 40%)	(*b*)	60,000	....	....	....	......	12,000,000	5,000,000	30
Alloyed aluminum, rolled {Aluminum 92%, Copper 8%}	16,000	28,000	....	....	....	......	......	......	15
Alloyed nickel, monel metal {Nickel 67%, Copper 28%}	40,000	75,000	....	....	....	......	25,000,000	9,500,000	
Concrete, 1 : 2 : 4 mix: {1 cement : 2 sand: 4 crushed stone}	....	200	....	2,000	....	......	2,000,000		
Timber: Yellow pine, small clear specimens	....	....	Parallel to grain 4,000; 900 across grain	Parallel to grain 6,000		Parallel to grain = 800	1,500,000		
Timber: White oak, small clear specimens	....	....	Parallel to grain 5,000; 2,000 across grain	Parallel to grain 9,000	....	......	2,000,000		
Limestone	....	Small	....	3,000 to 20,000	....	1,000 to 2,000	4,000,000 to 14,000,000		

(*a*) The yield-point, which is 2000 to 5000 lb. per sq. in. above the proportional limit, is the maximum useable compressive strength of a ductile material.
(*b*) Not well defined.
(*c*) Somewhat greater than the tensile strength since the material fails in tension when tested in shear.

cussion of the significance of the properties of structural materials is given in Chapter XV, and tables giving values of the properties of most of the structural materials are given in Chapter XVI.

In commercial testing of ductile material, such as structural steel, the four quantities usually found are the yield-point, the ultimate strength, the elongation in per cent and the reduction of area in per cent; the proportional limit, elastic limit and modulus of elasticity are seldom found in commercial tests.

The percentage of elongation is found by dividing the increase in length of a bar after rupture has occurred by the original length and multiplying by 100; and the percentage of reduction of area is found by dividing the difference between the areas of the ruptured and original sections by the area of the original section and multiplying by 100. The percentage of elongation and of reduction of area are measures of the ductility of a material (see Art 141 for further discussion).

The following facts should be noted from a study of Table 1:

(*a*) The shearing proportional limit for steel is about six-tenths (0.6) of the tensile proportional limit.

(*b*) The ultimate strengths of the more ductile steels are about twice as large as the yield-points.

(*c*) The tensile moduli of elasticity of all grades of steel are equal, the value of the modulus being 30,000,000 lb. per sq. in.; thus the stiffness of steel is constant, whereas the strength of steel varies greatly with the composition and treatment.

(*d*) The shearing moduli of elasticity of all grades of steel are equal, the value of the modulus being two-fifths of the tensile modulus; $E_s = \frac{2}{5} E_t = \frac{2}{5}\, 30{,}000{,}000 = 12{,}000{,}000$ lb. per sq. in.

(*e*) The maximum useable compressive strength of *ductile* material is the yield-point of the material.

(*f*) The compressive strength of brittle material (cast iron, concrete, stone, etc.) is greater than the tensile strength.

(*g*) The strength of timber varies greatly with the direction of the grain.

ILLUSTRATIVE PROBLEMS

Problem 10. A short concrete compression member is reinforced with 12 rods of steel $\frac{1}{2}$ in. in diameter arranged in a circle with a 5-in. radius as shown in Fig. 16. A uniform pressure or load is applied to each end surface of the member. The concrete outside of the reinforcing rod is used for protecting the rods in case of fire and is not assumed to resist any of the load.

If the load causes a compressive unit-stress of 600 lb. per sq. in. in the concrete, what is the unit-stress in the steel rods, and what is the total load carried by the member. The compressive moduli of elasticity of the concrete and steel are $E_1 = 2{,}000{,}000$ lb. per sq. in., and $E_2 = 30{,}000{,}000$ lb. per sq. in., respectively.

Solution.—Let the subscript 1 refer to concrete and the subscript 2 refer to the steel. Thus,

$$E_1 = \frac{s_1}{\epsilon_1} \quad \text{or} \quad s_1 = E_1 \epsilon_1 \quad . \quad (9)$$

and

$$E_2 = \frac{s_2}{\epsilon_2} \quad \text{or} \quad s_2 = E_2 \epsilon_2. \quad . \quad (10)$$

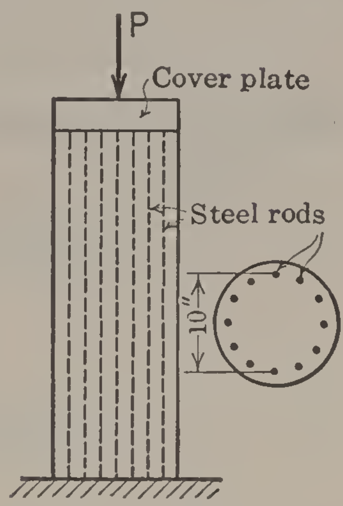


FIG. 16.—Concrete compression member reinforced with steel rods.

But since the concrete and the steel shorten the same amount, $\epsilon_1 = \epsilon_2$. Hence, dividing (9) by (10), we have

$$\frac{s_2}{s_1} = \frac{E_2}{E_1} = 15,$$

or

$$s_2 = 600 \times 15 = 9000 \text{ lb. per sq. in.}$$

Therefore, the compressive stress in the steel is 9000 lb. per sq. in. Thus, the stress in the steel increases with deformation 15 times as fast as does the stress in the concrete; that is, steel is much stiffer than is concrete.

The total load P, then, is

$$P = 9000 \times 12 \times \frac{\pi (\frac{1}{2})^2}{4} + 600 \times \frac{\pi (10)^2}{4}$$
$$= 68{,}300 \text{ lb.}$$

Problem 11.—A bar is l in. long, has a constant cross-sectional area of a sq. in., and weighs w lb. per ft. of length per sq. in. of cross-section. Find the total elongation of the bar when it is suspended from one end and is subjected to no downward load except its own weight. If the bar is made of steel and is 400 ft. long, calculate the total stretch of the bar. (A steel bar having a cross-sectional area of 1 sq. in. weighs 3.4 lb. per ft. of length.)

Solution.—The unit-stress, s_t, on a section at any distance y in. (Fig. 17) from the lower end expressed in lb. per sq. in. is

FIG. 17.—Bar stretched by its own weight.

$$s_t = \frac{awy}{12a} = \frac{wy}{12}$$

and the unit-deformation at this section is

$$\epsilon_t = \frac{s_t}{E_t} = \frac{wy}{12E_t}.$$

The elongation of the short length, dy in., of the bar along which the unit-elongation may be assumed to be constant is

$$de_t = \epsilon_t dy,$$

and the total elongation (in inches) in the length of l in. is

$$e_t = \int^l \epsilon_t dy = \int_0^l \frac{wy}{12E_t} dy$$

$$= \frac{w}{12E_t}\int_0^l y\,dy = \frac{w}{12E_t}\frac{l^2}{2}$$

$$= \frac{3.4 \times (400 \times 12)^2}{12 \times 30{,}000{,}000 \times 2}$$

$= 0.107$ in. $=$ elongation of steel bar 400 ft. long.

PROBLEMS

12. It is specified that a steel rod 40 in. long is to be subjected to a unit-stress not greater than 10,000 lb. per sq. in. and to be elongated not more than 0.01 in. when resisting a tensile axial load of 20,000 lb. Determine the cross-sectional area required to satisfy each of the specifications and state which requirement governs the design.

13. In a tension test of a steel bar 0.499 in. in diameter the elongation was measured over a gage length of 2 in. Successive readings of the load and of the extensometer were as given herewith. Determine the elastic limit, the proportional limit and the yield-point of the material.

Load in Pounds	Reading of Extensometer in Inches
500	0.0
1500	0.0005
500	0.0
3000	0.0010
500	0.0
4490	0.0020
500	0.0001
5980	0.0025
500	0.0002
7510	0.0030
500	0.0005
8630	0.0035
500	0.0010
9500	0.0010
500	
9600	0.0130

14. Two blocks, each 4 in. by 8 in. by 40 in., are bolted together to form a compression member as shown in Fig. 18. A pressure is applied to the top surface of the member causing both blocks to shorten the same amount.

If one of the blocks is made of gray cast iron and the other of oak, find the total load P when the unit-stress in the cast iron is 25,000 lb. per sq. in.

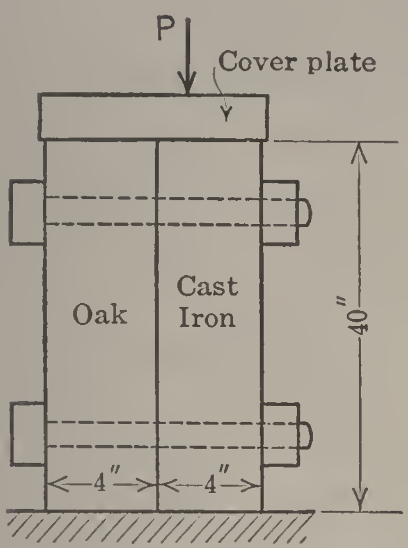


FIG. 18.—Two-material compression block.

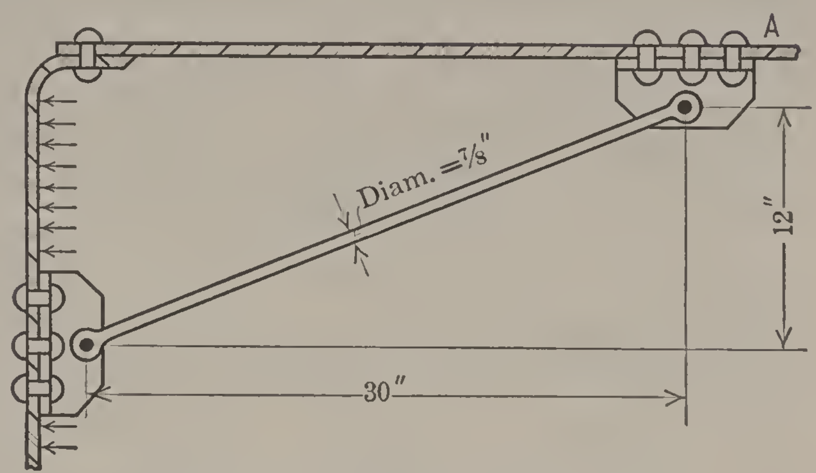


FIG. 19.—Brace in steam boiler.

15. Two wires, one of steel for which E_t is found to be 14,500 tons per sq. in., and one of copper for which E_t is found to be 7500 tons per sq. in., have the same length and carry equal axial loads. The copper wire has a diameter of 0.03 in. If each wire elongates the same amount, what is the diameter of the steel wire? *Ans.* $d = 0.0155$ in.

16. Will a permanent set be caused in a wrought-iron bar $\frac{7}{8}$ in. in diameter and 3 ft. long when subjected to an axial tensile load of 24,000 lb.?

17. A steel piano wire with a constant cross-section of 0.00038 sq. in. has an elastic limit of 100,000 lb. per sq. in. If the wire is used to let down a 30-lb. body from the top of a building 500 ft. high, what must be the original length of the wire if the body just touches the ground when suspended by the wire? (Steel weighs 0.28 lb. per cu. in.)

18. The boiler brace shown in Fig. 19 resists the pressure on an area of 80 sq. in. of the boiler-head. (*a*) If the steam pressure in the boiler is 120 lb. per sq. in., what is the tensile unit-stress in the brace? (*b*) If the brace is made of steel, how much does it elongate; assume that the rod is of constant diameter from pin to pin. (*c*) What is the shearing unit-stress in the three rivets at A if the diameter of each rivet is $\frac{3}{4}$ in.?
Ans. (*a*) 16,900 lb. per sq. in., (*b*) $e = 0.0182$ in., (*c*) $s_s = 6780$ lb. per sq. in.

19. How long must be the bar described in Problem 12 in order to cause a stress in the bar equal to the proportional limit of the material? The bar is made of structural steel.

7. Working Stress.—A *working stress* or an *allowable stress* for a material is the maximum unit-stress that is considered to be safe for the material when the material is resisting the loads that are assumed to be applied to it in service. The values of allowable stresses that are commonly used for various materials when resisting various types of loading have been determined largely from the results of tests of the materials and from the accumulated experience obtained in the construction and use of structures and machines under service conditions; Table 2 gives commonly used values of working stresses for several materials when the material is subjected to static loads (see Table X of Chapter XVI for a more extensive table of values).

The allowable working stress in a member is always considerably less than the ultimate strength of the material of the member. For brittle materials (materials that do not have yield-points), such as concrete, cast iron, high carbon steel, etc., the working stress is frequently taken as a certain proportion [5] of the ultimate strength of the material. For ductile materials, such as structural steel, wrought iron, etc., the working stress may be taken to be either a certain proportion [5] of the ultimate strength or a certain (but different) proportion of the yield-point. In general, the working stress for a ductile material must be less than the yield-point (and usually considerably less) since most structures would not fulfill their function if the members of the structure were stressed beyond their yield-points and thus became permanently deformed, although they might be safe against rupture or collapse. (For a further discussion of this point see Art. 138.)

Need of a Margin of Strength.—The need for selecting working stresses considerably less than the ultimate strength (or less than the yield-point) arises from (1) the uncertainties as to the properties of the materials used, (2) the uncertainties as to the loads to be resisted by the structure or machine as a whole and also by the various members of the structure, and (3) the uncertainties in the methods of calculating the stresses in the members. The more these uncertainties are reduced the higher the working stresses may be. And, if the working stresses are increased, the amount

[5] The term *factor of safety* of a member has been widely used to denote the ratio of the ultimate strength of the material of the member to the working stress used in designing the member. The use of this term and the idea it conveys, however, is now generally conceded to be misleading and hence undesirable. For example, if a factor of safety of 4 is used with structural steel in selecting a working stress, a structure designed on the basis of this working stress would *not* resist loads *four* times as great as the loads assumed to act on the structure and to produce the working stresses. For further discussion, see Arts. 138 and 139.

of materials purchased (and hence the cost of construction) will decrease.

In building constructions, working stresses have become, to a large degree, standardized. Thus, the building laws of most large

TABLE 2

VALUES OF WORKING STRESSES FOR USE WITH STATIC LOADS
(See also Table X of Chapter XVI)
Values of stresses are expressed in lb. per sq. in.

Material	Type of Stress			Proportion of Ultimate Strength
	Direct Tension	Direct Compression	Direct Shear	
Structural steel......	16,000–18,000	14,000–16,000	8,000–10,000	0.25 for tension * 0.20 for shear
Wrought iron.......	15,000	12,000	8000	0.25 for tension * 0.20 for shear
Gray cast iron......	3,000	15,000	....	0.15 for tension 0.20 for compression
Timber (yellow pine).	....	1000 with the grain 250 across the grain	....	0.15 with the grain
Portland cement concrete; 1 : 2 : 4 mix.	....	450	....	0.22

* For ductile material, such as structural steel, w ought iron, etc., the working stress is usually thought of as a proportion of the yield-point since for such material the yield-point, rather than the ultimate strength, is the maximum useable strength of the material (see Art. 138); if this were done, the values would be 0.5 for tension and 0.4 for shear, since the yield-points of structural steel and wrought iron are approximately one-half of the ultimate strengths.

cities specify the maximum allowable working stresses to be used for the common structural materials (see Table X of Chapter XVI). For example, a working stress of 16,000 lb. per sq. in.[6] is specified

[6] When structural steel is purchased according to standard specifications, and subjected to rigid inspection, etc., working stresses higher than 16,000 lb. per sq. in. (18,000 to 20,000 lb. per sq. in.) are sometimes allowed n tension and flexural members; for steel *compressive* members, however, the working stress is usually specified to be less than 16,000 lb. per sq. in. (usually from 12,000 to 14,000 lb. per sq. in.; see Art. 139).

by most building codes for tension members and for beams, etc., when made of rolled structural steel. In machine construction standardization of working stresses is more difficult than in structural work since the variety of materials used is greater and the conditions of service are more uncertain. However, a manufacturing firm that builds machines of certain types is able to standardize the working stresses for the design and construction of those machines.

The decrease in cost of construction accompanying the use of higher working stresses acts as a stimulus to reduce the uncertainties referred to above. Thus, the methods of manufacture and treatment of materials are being refined; tests and inspection of materials are becoming more rigid and more generally used; knowledge of the properties of materials and of the methods of calculating stresses in the members is becoming more extended and more generally applied, and experience and tests are gradually yielding more definite information as to the loads that various types of structures and machines are subjected to in service. The standardization of working stresses by law and by practice, of course, helps greatly to safeguard the public, who use engineering structures such as buildings, bridges, locomotives, elevators, etc., against unscrupulous or incompetent engineering design.

Factors Affecting Values of Working Stresses.—From Table 2 it will be noted that the working stresses for cast iron and timber are relatively small in relation to the ultimate strength of the material, the reason being that these materials are less uniform in structure than steel. For example, cast iron may contain blow holes, and initial stresses due to uneven cooling; timber may contain knots, pitch pockets, cracks, etc., and its strength is influenced markedly by its moisture content. Further, for a ductile material such as structural steel working stresses are relatively larger in relation to the ultimate strength than for a brittle material such as hard steel, cast iron, etc. The reason for this is that a ductile material deforms or yields if subjected to unexpected overloads or if high localized stresses occur at some portion of a member (and such stresses always do occur); the yielding distributes some of the excess stress to the surrounding materials and hence tends to prevent the member from breaking. (See Arts. 138 and 143 for further discussion.)

The working stress for any one material is, in general, smaller when the material resists impact and repeated loads than when it resists static loads; the reasons are that an impact load is in general less definitely known and it causes larger stresses and strains than does the same load when applied gradually, and localized stresses and non-homogeneity of material have a much greater influence on the strength of the member when the loads are repeated than when applied but once as a static load (see Art. 119 of Chapter XIII and Art. 131 of Chapter XIV for further discussion).

PROBLEMS

20. When a force, P, of 196 lb. is applied to the bell-crank shown in Fig. 20, the bearing pressure normal to the rubbing surfaces of the friction clutch is 15 lb. per sq. in. If the coefficient of friction for the two surfaces is $\frac{1}{4}$, what is the maximum shearing unit-stress that can be developed in the rectangular key if its dimensions are 1 in. wide, $\frac{5}{8}$ in. deep and 4 in. long? What is the ratio of this stress to the shearing yield-point of the material if the key is made of structural steel? (Tensile yield-point $= \frac{1}{2}$ tensile ultimate strength; shearing yield-point $= 0.6$ tensile yield-point.)

Ans. $s_s = 3710$ lb. per sq. in.; 0.19.

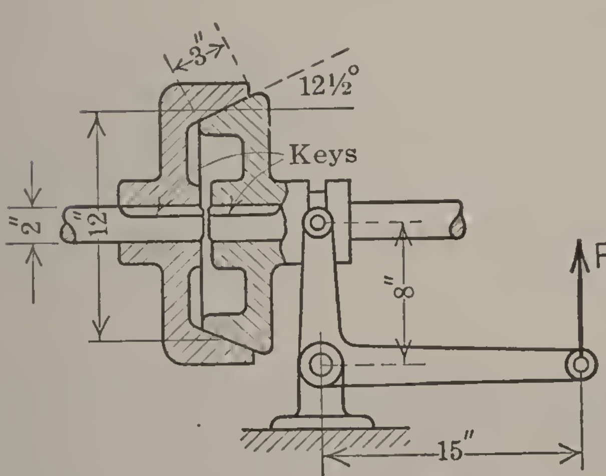


FIG. 20.—Shear in key of clutch.

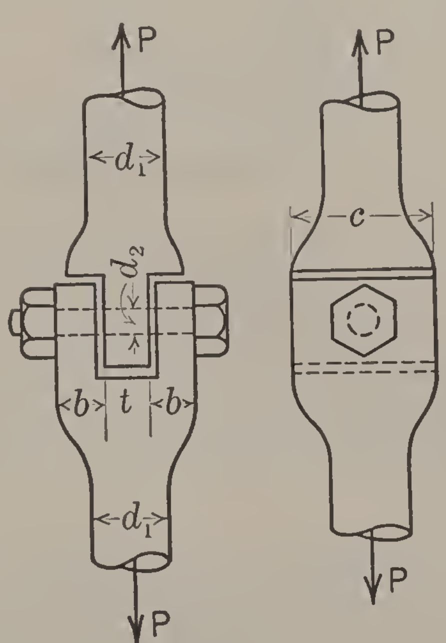


FIG. 21.—Two members connected by pin.

21. In the joint shown in Fig. 21, $d_1 = 2\frac{1}{2}$ in., $d_2 = 1\frac{1}{2}$ in., $t = 2$ in., $b = 1$ in., and $c = 5$ in. All of the material is structural steel. If the load P is 50,000 lb. and if the working stress is specified to be not greater than 0.3 of the yield-point, does the joint satisfy the specification? (For values of yield-points, see statement in Prob. 20.)

22. A hollow gray cast-iron compression member supports loads as shown in Fig. 22. The member is prevented from bending by lateral supports (not shown). If the outside diameter is 20 in., what should be the inside diameter if the working stress is one-eighth ($\frac{1}{8}$) of the ultimate strength?

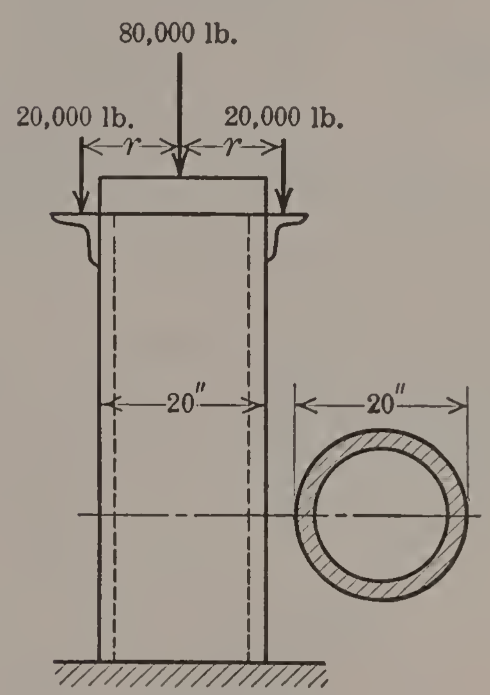


FIG. 22.—Hollow compression member.

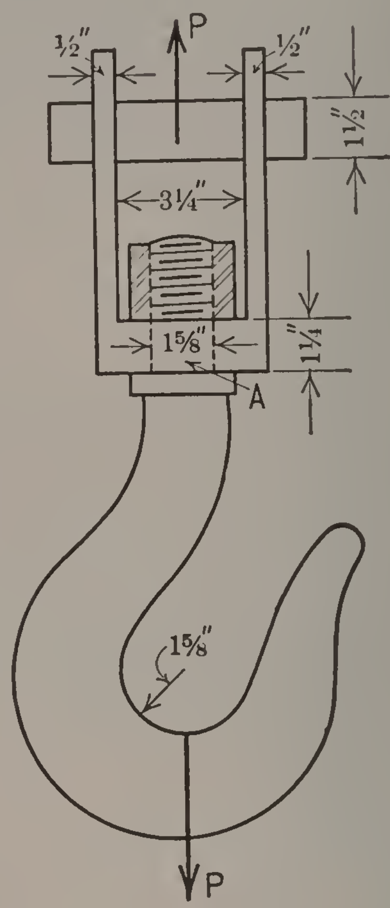


FIG. 23.—Crane hook.

23. The capacity of the hook and yoke shown in Fig. 23 is 10 tons. If the material has a yield-point of 40,000 lb. per sq. in. in tension and six-tenths as large in shear, what are the ratios of the tensile stress at section *A*, and of the shearing stress in the pin to the corresponding yield-points when the hook is subjected to its maximum load? *Ans.* 0.24; 0.34.

24. Figure 24 represents a lower panel point of a pin-connected Pratt truss. The post and the three eye-bars are made of structural steel having an ultimate strength of 65,000 lb. per sq. in. and a yield-point equal to one-half the ultimate strength. Find the area of each of the eye-bars using a working stress equal to 0.4 of the yield-point.

Ans. $a_P = 2.18$ sq. in., $a_Q = 3.08$ sq. in.

25. Figure 25 represents two bars of diameter d_1 joined by means of a cotter pin, the axes of the bars being in the same straight line. The upper bar is enlarged at its lower end to form a hollow cylindrical socket which fits over the enlarged upper end of the lower bar. The two bars and the cotter pin are made of structural steel. From the following data find the ratios of the tensile stress in the two bars, of the tensile stress in the socket, of the shearing stress in cotter pin, and of the bearing stress on the cotter pin

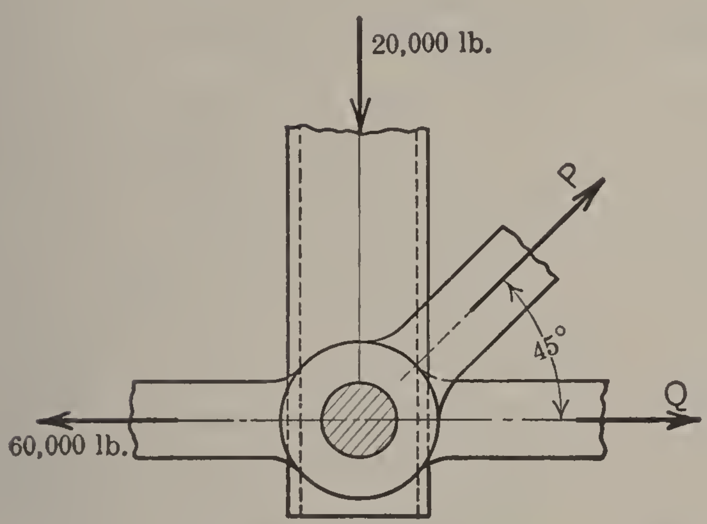


FIG. 24.—Stresses in pin-connected truss.

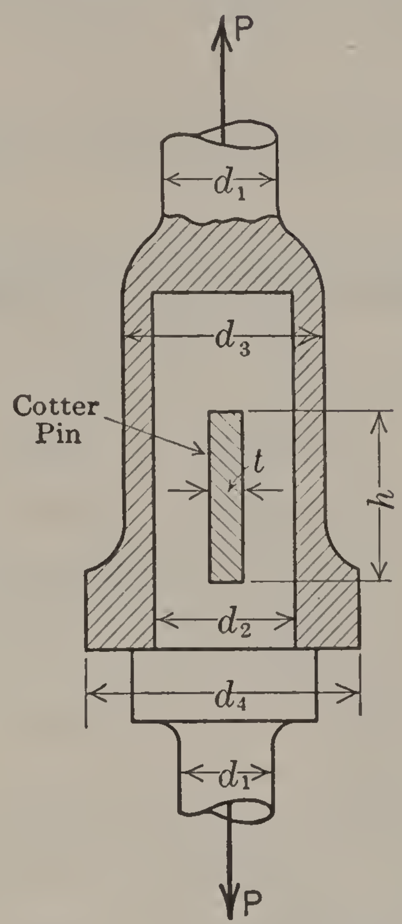


FIG. 25.—Socket joint and cotter pin.

to the corresponding ultimate strengths of the material. The ultimate bearing stress may be assumed to be 80,000 lb. per sq. in.

$P=60{,}000$ lb.; $d_1=2$ in.; $d_2=2\frac{1}{2}$ in., $d_3=3\frac{1}{2}$ in., $d_4=5\frac{1}{2}$ in., $t=1$ in., and $h=3$ in.

CHAPTER II

THIN-WALLED CYLINDERS AND SPHERES. RIVETED JOINTS

8. Stresses in Thin-walled Cylinders and Spheres.—A thin-walled cylinder or sphere is one in which the thickness of the wall or shell is small in comparison with the diameter of the vessel. When this condition is satisfied the stress in the shell due to an internal fluid pressure may be considered to be uniformly distributed on the cross-sectional area of the shell without introducing serious errors in the calculation of the stress. Boilers, tanks, water and steam pipes usually may be treated as thin-walled cylinders, and hoops and tires that are shrunk on wheels may also usually be so treated. The cylinders of large guns and of some pipe, such as the pipes to hydraulic forging presses, etc., have thick walls and the stress in the walls cannot be assumed to be uniformly distributed without introducing a large error in the calculated unit-stress.

The problem here considered is to determine the relation between the internal pressure in a closed thin-walled cylinder, the diameter of the cylinder, the thickness of the shell, and the intensity of stress in the shell (*a*) on a longitudinal section and (*b*) on a transverse section.

Stress on a Longitudinal Section.—Fig. 26(*a*) represents a portion of a thin-walled cylinder that is subjected to an internal fluid pressure of intensity R, the length of the portion being l. Let the diameter of the cylinder be denoted by D, the thickness of the shell by t, and the intensity of the tensile stress in the shell by s_t.

The pressure of the fluid on the internal surface of the cylinder at any point is normal to the surface at that point, as indicated in Fig. 26(*a*), and these internal pressures tend to rupture the cylinder on a longitudinal section (at AB and EF, Fig. 26*a*), the resultant pressure (or load) on one-half of the shell (Fig. 26*b*) being resisted, and held in equilibrium, by the total stresses P, P exerted by the other half of the shell at the areas AB and EF. Hence from equilibrium we have,

Resultant horizontal pressure or load = total resisting stress.

An expression for the resultant horizontal pressure on the half-shell may be found as follows: The resultant horizontal pressure on the semi-cylindrical area, a, of the half-shell (Fig. 26*b*) is the sum of the horizontal components of the pressures on the ele-

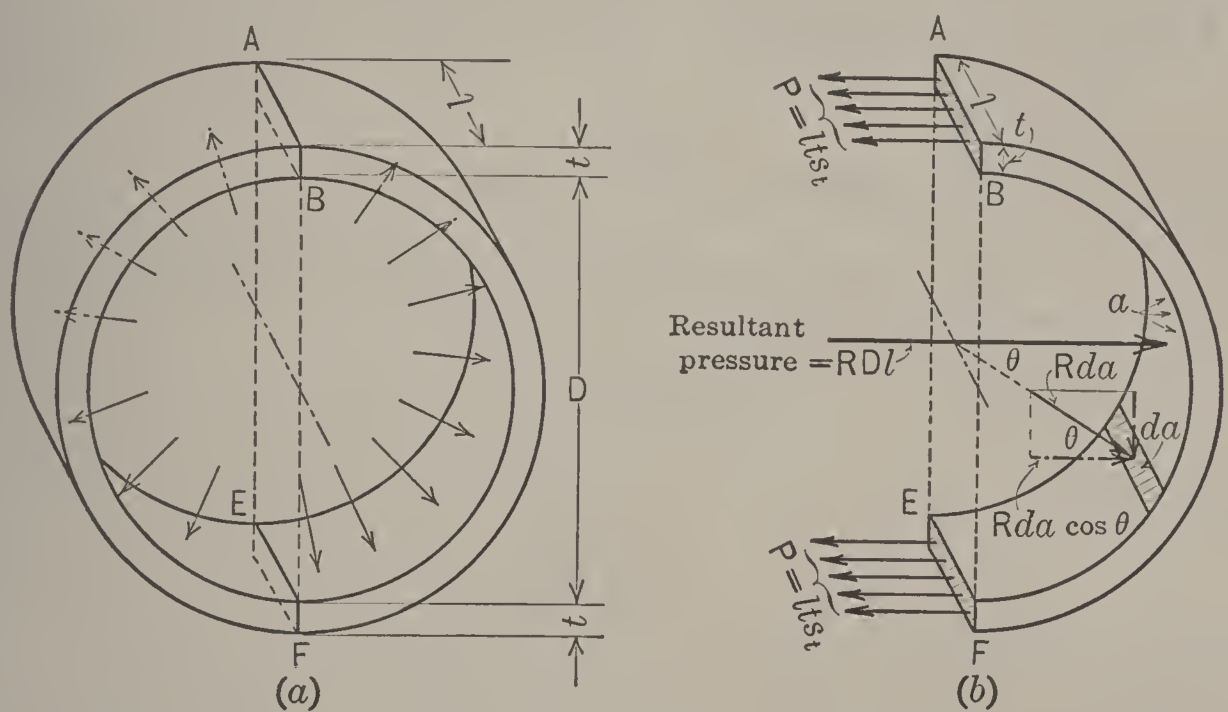


FIG. 26.—Stress on longitudinal section of thin-walled cylinder.

mentary areas; the pressure on an elementary area da is Rda and its horizontal component is $Rda \cos \theta$, and hence the resultant horizontal pressure is equal to $\int Rda \cos \theta$, which may be written, $R \int da \cos \theta$ since the pressure R is the same at all points in the semi-cylindrical area, a. But $da \cos \theta$ is the area formed by projecting the area da on a vertical plane and hence $\int da \cos \theta$ is the area formed by projecting the semi-cylindrical area on a vertical plane, and is, therefore, equal to Dl. Thus, the resultant horizontal pressure is Rdl.

Further, since the shell is thin, the total resisting stress, $2P$, may be assumed to be distributed uniformly over each of the two areas, and hence $2P = 2a_t s_t = 2lts_t$. Therefore, the above equation becomes

$$RDl = 2lts_t.$$

Whence,

$$s_t = \frac{RD}{2t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Stress in Transverse Section.—The total pressure of the fluid against the end of the cylinder must be resisted by the total stress on a transverse section of the cylinder as indicated in Fig. 27. Now the total pressure against the end of the cylinder is $R\frac{\pi D^2}{4}$, and the total resisting stress is $\pi Dt \cdot s_t$. Hence from the condition of equilibrium we have,

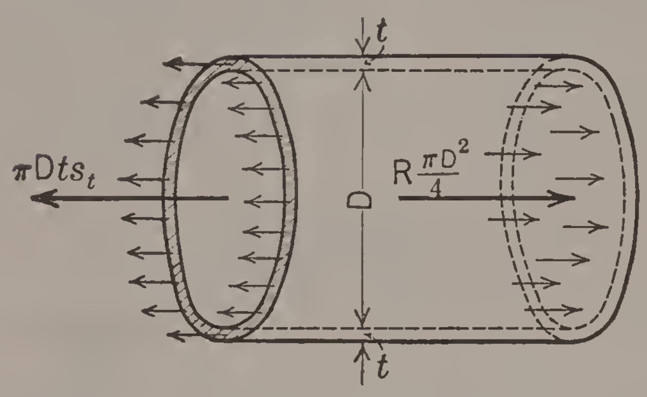


FIG. 27.—Stress on traverse section of thin-walled cylinder.

$$R\frac{\pi D^2}{4} = \pi Dt \; s_t,$$

whence,

$$s_t = \frac{RD}{4t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In (11) and (12) R and s_t must be expressed in the same units (usually pounds per square inch), and D and t must be expressed in the same units (usually inches).

A comparison of equations (11) and (12) shows that the intensity of stress on a longitudinal section of a thin-walled cylinder due to an internal fluid pressure is twice as great as that on a transverse section of the same cylinder. This fact explains why the riveted joint connecting the plates of a boiler along a longitudinal seam requires more rivets than that along a transverse seam.

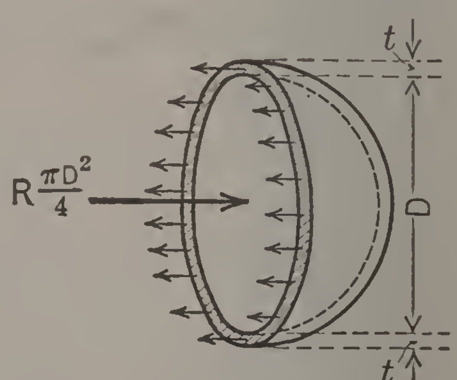


FIG. 28.—Stress in thin-walled sphere.

Equation (12) may be used also to find the unit-stress in a thin-walled sphere due to internal fluid pressure as will be evident when the method used above in determining the stress in a thin-walled cylinder is applied to a thin-walled sphere as represented in Fig. 28.

PROBLEMS

26. A boiler is subjected to an internal pressure of 80 lb. per sq. in. and is 48 in. in diameter. (*a*) What is the total stress acting on (or transmitted across) each of the two longitudinal cross-sectional areas of the plate or shell in a length of 3 in. along the boiler? (*b*) What is the unit-stress developed in the shell if the thickness of the shell is $\frac{1}{2}$ in.?

27. A standpipe 6 ft. in diameter is 60 ft. high. When it is full of water, what is the circumferential unit-stress in the plate at the bottom of the standpipe if the thickness of the plate is $\frac{3}{8}$ in.? Hint: Since water weighs 62.5 lb. per cu. ft., the pressure (in all directions) at the bottom of the standpipe is 62.5×60 lb. per sq. ft. *Ans.* $s = 2500$ lb. per sq. in.

28. The pressure in the cylinder of a steam-engine (Fig. 29) is 120 lb. per sq. in. and the internal diameter, D, of the cylinder is 14 in. How many $\frac{3}{4}$-in. bolts are required for strength if the tensile unit-stress is not to exceed 8000 lb. per sq. in.? What should be the thickness, t, of the walls of the cast-iron cylinder to satisfy the requirement for strength if the allowable tensile stress is 8000 lb. per sq. in.? Note.—The allowable stresses are taken low due to the fact that the load is applied with more or less impact. Further, the requirement for strength in many problems is not the governing requirement. In this problem, for example, the bolts should be large enough to prevent a workman, with ordinary tools, from twisting the heads off. Further, the requirement for tightness of the joint may determine the number of bolts. Similarly, the thickness of the wall may be influenced by the considerations of heat loss or of ease and reliability in casting, etc.

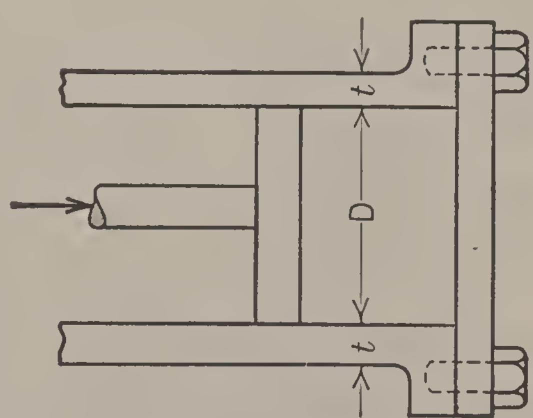


Fig 29.—Steam cylinder and piston.

29. Water pipes are frequently made of cast iron. According to specifications a pipe 18 in. in diameter must have a wall thickness of 0.87 in. and must resist an internal pressure of 300 lb. per sq. in. What circumferential unit-stress is developed in the pipe? *Ans.* $s = 3100$ lb. per sq. in.

30. The steel tire for a locomotive driving-wheel has an internal diameter $\frac{1}{1500} \cdot d$ less than that of the wheel on which the tire is to be shrunk, where d is the diameter of the wheel. The value of d is 60 in. and the value of t, the thickness of the tire, is $\frac{3}{4}$ in. If it is assumed that after the tire is shrunk on the wheel the diameter of the wheel is not changed by the pressure of the tire, find (1) the elongation of the tire, (2) the tensile unit-stress (hoop-tension) in the tire, and (3) the intensity of the pressure of the tire on the wheel. *Ans.* (1) $e = 0.126$ in., (2) $s = 20{,}000$ lb. per sq. in., (3) $R = 500$ lb. per sq. in.

Riveted Joints

9. Introduction.—Riveted joints are important structural elements in buildings, bridges, cranes, etc., and also in pressure vessels such as boilers, tanks, water pipes, etc.

The load causing the stresses in the various parts of a riveted joint (shearing stress in the rivets, bearing stress on the rivets, tensile stress in the plate, see Fig. 32) is usually assumed to act through the centroid of the area on which the stresses occur, and hence the stress on the resisting area is assumed to be uniformly distributed. This assumption, however, must be regarded as a rough approximation only, since, as discussed in Art. 13, the load on a riveted joint is seldom distributed evenly to the rivets; however the design of the simpler types of joints are usually based on this assumption. The stresses in riveted joints produced by eccentric loads are discussed in Chapter X, but eccentric loads, whenever possible, should be avoided.

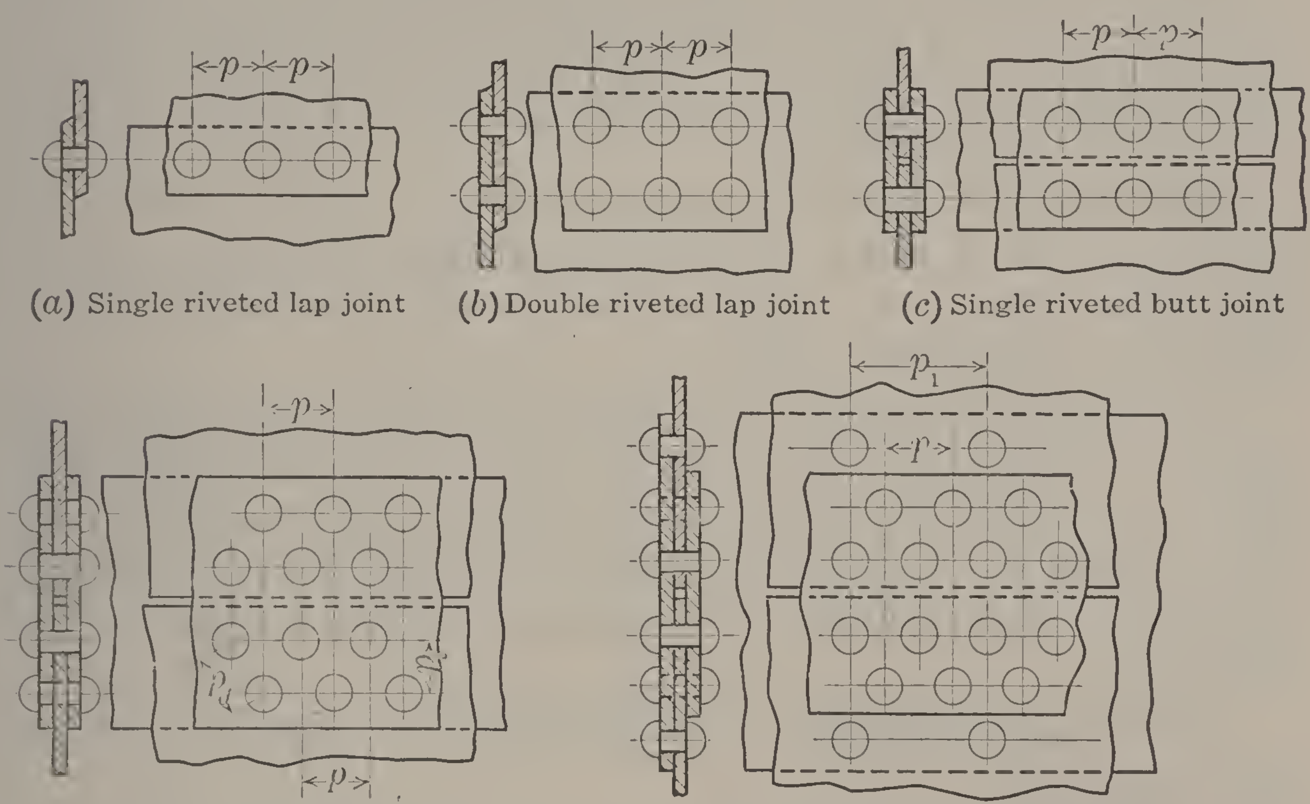
According to the above assumption the stresses in a riveted joint may be found from the equation developed in Chapter I for a central load; namely, $P = as$, in which P is the central load transmited from one to the other of two plates that are connected by the rivets, and s is any one of the various unit-stresses (shearing, bearing, tension, etc.) that may be the cause of the failure of the joint (as discussed in Art. 11) and a is the area on which the stress is distributed.

10. Types of Riveted Joints. Definitions.—It is convenient to divide riveted joints into two general groups; (1) structural joints used in connecting members in bridges, buildings, cranes, and other structures, and (2) boiler, tank, or pipe joints used in connecting plates in various types of pressure vessels. In structural joints, strength is the main requirement whereas in boiler and pipe joints tightness in addition to strength must be considered.

Two types of joints are widely used for both the groups mentioned above; namely, lap joints and butt joints (see Fig. 30), and both lap and butt joints may be single riveted, double riveted, triple riveted, etc., according as one, two, three, etc., rows of rivets pierce each of the two plates that are connected. Further, a butt joint may have two cover plates (or straps) or only one cover plate. Again, both lap and butt joints may have the rivets arranged in the form of chain riveting (Fig. 30*b* and 30*c*) or in the form of staggered riveting (Fig. 30*d* and 30*e*).

Definitions.—The terms defined below will be used frequently in the subsequent articles.

The *pitch* of a row of rivets is the distance between the centers



(*a*) Single riveted lap joint (*b*) Double riveted lap joint (*c*) Single riveted butt joint

(*d*) Double riveted butt joint (*e*) Triple riveted butt joint

FIG. 30.—Types of riveted joint.

of any two adjacent rivets in the row. The pitch is not necessarily the same for all rows. Thus, in Fig. 30*e* the pitch for the inner rows is denoted by p and for the other row by p_1 in which p_1 equals $2p$.

The *transverse pitch* is the distance between the center lines of two rows of rivets, denoted by p_t in Fig. 30(*d*), and the *diagonal pitch* in staggered riveting is the distance from the center of a rivet in one row to that of the nearest rivet in the next row, denoted by p_d in Fig. 30(*d*).

The *length of a repeating group* of rivets is used in connection with boiler and pipe joints to denote the shortest distance along the joint that includes a characteristic group of rivets which recurs along the length of the joint. The length of the repeating group is usually equal to the pitch, the maximum pitch being used when the pitches of all rows are not the same. In computations it is convenient to use the force transmitted through the repeating group of rivets rather than that transmitted through the entire length of the joint.

The *margin* of a plate in a riveted joint is that part of the plate between the edge of the plate and the center line of the nearest row of rivets.

A *gusset plate* or *splice plate* is a plate used in structural joints to form part of the joint in connecting two or more members of a structure (see Fig. 33*b*). It corresponds to a cover plate or strap in a boiler or pipe butt joint.

Auxiliary piece is a term used to denote any plate or other piece which is used in addition to the rivets and the main plates or members to form the joint. Thus, a gusset plate or a column bracket (Fig. 35) is an auxiliary piece.

The term *efficiency* of a riveted joint is used in connection with boiler and pipe joints to denote the ratio of the strength of the joint to the strength of the solid plate. It is customary to use the working or allowable strengths (not allowable *unit-stresses*) rather than the maximum or ultimate strengths in calculating the efficiency.

11. Modes of Failure.—A riveted joint may fail in any of the following ways:

1. Shearing of the rivets as indicated in Fig. 31(*a*).

2. Rupturing of the plate on a section through a line of rivets or on a section along a diagonal pitch as indicated in Fig. 31(*b*).

3. Crushing of the rivets (or of the plate) due to the pressure of the plate on the side of the rivet (or of the rivet on the plate) as indicated in Fig. 31(*c*). This pressure is called the bearing pressure.

4. Shearing of the plate in the margin or tearing of the plate in the margin, as indicated in Fig. 31(*d*). A marginal failure is usually a combination of these two actions.

In joints having several rows of rivets the failure may be a combination of the above failures, as for example the rupturing of the plate along one row of rivets accompanied by the shearing of the rivets in another row.

The width of the margin can be fixed arbitrarily (except as it affects calking of a boiler joint to secure tightness) and experience has shown that if the width of the margin is $1\frac{1}{2}$ to 2 times the diameter of the rivet the joint will not fail in the margin. Further, experience has shown that failure along a diagonal will not occur if the transverse pitch is not less than $1\frac{3}{4}$ times the diamter of the rivet. It is evident therefore that if these conditions are satisfied the only stresses that need investigation in a riveted joint are (1) the shearing stress in the rivets, (2) the tensile stress in the plate on a section through a row of rivets, and (3) the bearing stress on the rivet (or on the plate).

12. Stresses in Riveted Joints.—As noted in Art. 9 the equation $P = as$ is used in the analysis of the stresses in a riveted joint

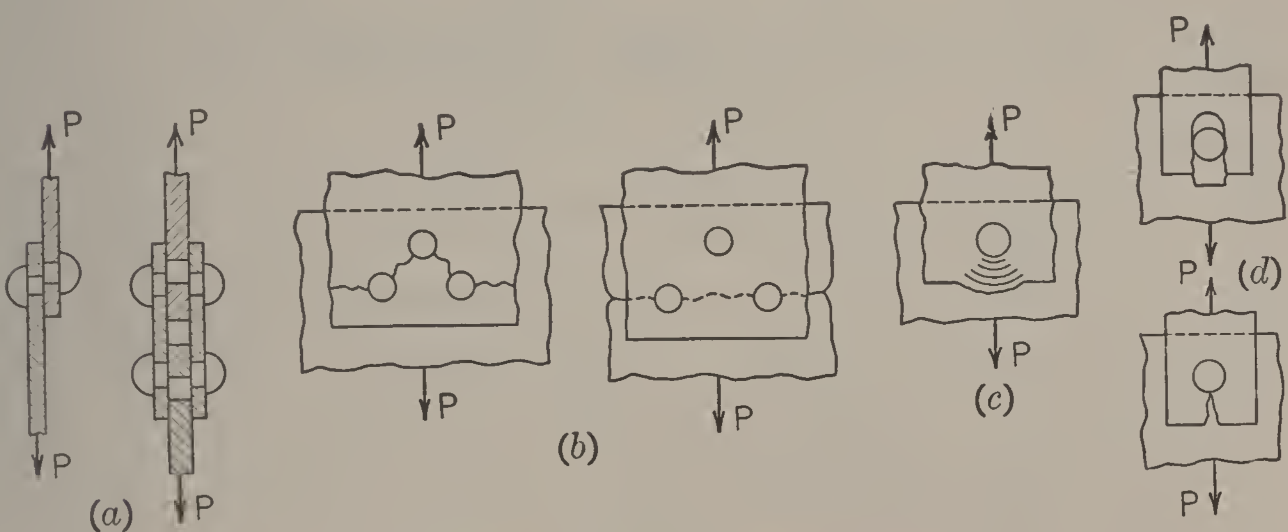


FIG. 31.—Modes of failure of riveted joints.

in which the load on the joint is assumed to act through the centroid of the rivet areas, the stress on each area then being assumed to be uniformly distributed.

In the equation $P = as$ as applied to riveted joints,

P, for a structural joint, is the force that is transmitted from one member of a structure to another member, and, for a boiler or pipe joint it is usually the force transmitted through a repeating group of rivets.

The kind of stress s will be denoted by a subscript. Thus, s_s denotes the shearing unit-stress in the rivets; s_t, the tensile unit-stress in the plate; and s_b the bearing unit-stress on the rivet. Similarly a_s denotes the shearing area on which s_s occurs, a_t the tensile area on which s_t occurs; and a_b the bearing area on which s_b occurs.

Other quantities used are defined as follows:

t is the thickness of the main plates or members;
d is the diameter of the rivets;
p is the pitch of the rivets.

To illustrate the method of determining the stresses in a riveted joint one particular joint will here be used; the method, however, is the same for all joints. Fig. 32(a) represents a double-riveted two-strap butt joint, in which it is desired to find the shearing,

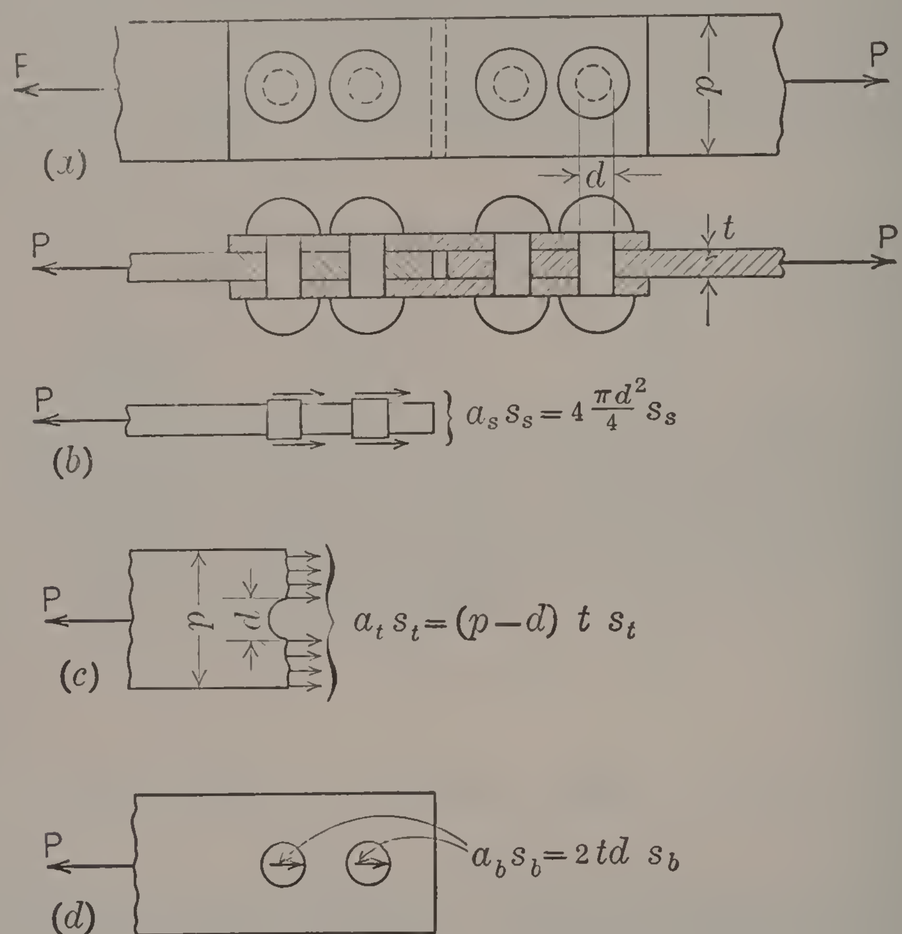


FIG. 32.—Stresses in riveted joint.

tensile and bearing unit-stresses in terms of the load P and the dimensions, t, d and p.

Shearing Unit-stress.—The load P is held in equilibrium, as shown in Fig. 32(b), by the total resisting shearing stress, which is the product of the shearing unit-stress s_s and the area a_s on which the shearing stress occurs. And since each of the two rivets has two shearing areas, a_s is equal to $\frac{4\pi d^2}{4}$. Hence

$$P = a_s s_s = 4\frac{\pi d^2}{4} s_s \quad \text{or} \quad s_s = \frac{P}{\pi d^2}. \quad . \quad . \quad . \quad . \quad (13)$$

It should be noted that if the above joint were a *lap* joint, each rivet would have only one shearing area, and hence the total area would be $2\frac{\pi d^2}{4}$ instead of $4\frac{\pi d^2}{4}$.

Tensile Unit-stress.—The total tensile resisting stress, $a_t s_t$, in the plate which holds the load P in equilibrium (see Fig. 32*c*) is the product of the area a_t, or $(p-d)t$ and the tensile unit-stress s_t. Hence

$$P = a_t s_t = (p-d)ts_t \quad \text{or} \quad s_t = \frac{P}{(p-d)t}. \quad . \quad . \quad (14)$$

Bearing Unit-stress.—The total bearing stress of the plate on the rivets which resists the load P (Fig. 32*d*) is the product of the bearing unit-stress s_b and the bearing area a_b on which the stress is assumed to be uniformly distributed. The area of contact between the rivet and plate is a semi-cylindrical area and although the intensity of the pressure and the direction of the pressure probably vary greatly over the area, the manner of variation is unknown. Only the component pressure parallel to the load P resists this load, and it is usually assumed that this component pressure is uniformly distributed over an area equal to the projection of the semi-cylindrical area of contact on a plane perpendicular to the direction of P, that is, an area equal to the product of the diameter of the rivet and the thickness of the plate. Hence, as indicated in Fig. 32*d*,

$$P = a_b s_b = 2td\ s_b \quad \text{or} \quad s_b = \frac{P}{2td}. \quad . \quad . \quad . \quad . \quad (15)$$

As noted above, the load P in boiler joints usually is taken as the load that is resisted by a repeating group of rivets and in structural joints it is the total force transmitted from one member to another.

ILLUSTRATIVE PROBLEMS

Problem 31. The members of the Fink truss shown in Fig. 33(*a*) are connected by riveted joints. The arrangement of the joint at C is shown in Fig. 33(*b*). The stress in members BC and DC are found to be 6930 lb. compression and 6930 lb. tension respectively. The rivets are $\frac{3}{4}$ in. in diameter. Find the shearing unit-stress in the rivets connecting the members BC and DC to the gusset plate, and in the rivets connecting the gusset plate to the lower chord $ACEG$. Also find the bearing unit-stress of the members BC and DC on the rivets.

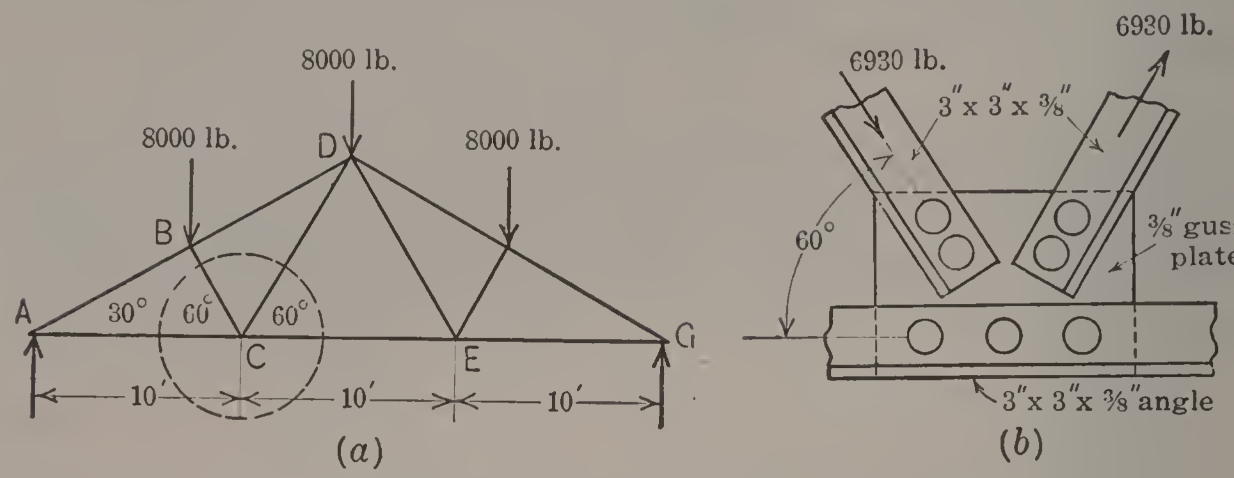


FIG. 33.—Riveted joint in truss.

Solution.—The shearing unit-stress in the rivets connecting members BC and DC to the gusset plate is

$$s_s = \frac{P}{a_s} = \frac{6930}{2 \times 0.44} = 7870 \text{ lb. per sq. in.,}$$

the area of cross-section of a $\frac{3}{4}$-in. rivet being 0.44 sq. in.

The bearing unit-stress of the members BC and DC (or of the gusset plate) against the rivets is

$$s_b = \frac{P}{a_b} = \frac{6930}{2 \times \frac{3}{4} \times \frac{3}{8}} = 12{,}300 \text{ lb. per sq. in.}$$

The shearing unit-stress in the rivets that connect the gusset plate to the lower chord is

$$s_s = \frac{P}{a_s} = \frac{2 \times 6930 \cos 60°}{3 \times 0.44} = \frac{6930}{1.32} = 5240 \text{ lb. per sq. in.}$$

The bearing unit-stress of the lower chord on the three rivets is

$$s_b = \frac{P}{a_b} = \frac{6930}{3 \times \frac{3}{4} \times \frac{3}{8}} = 8210 \text{ lb. per sq. in.}$$

Problem 32.—A boiler having a diameter, D, of 72 in. is designed to resist an internal steam pressure, R, of 120 lb. per sq. in. The longitudinal joint is a double-riveted lap joint. The thickness, t, of the plates is $\frac{7}{16}$ in., the diameter, d, of the rivets is $\frac{7}{8}$ in., and the pitch, p, is $3\frac{1}{2}$ in. Find the shearing, tensile, and bearing unit-stresses in the joint. Also calculate the efficiency of the joint, assuming the allowable shearing, tensile, and bearing unit-stresses (in lb. per sq. in.) to have the following values: $s_s = 10{,}000$, $s_t = 15{,}000$, $s_b = 22{,}500$ (see Art. 14).

Solution.—According to Art. 8 and as indicated in Fig. 34(a) the force P transmitted through the joint for a pitch distance p of $3\frac{1}{2}$ in. along the joint, that is, through a repeating group of rivets (two rivets) is

$$P = \tfrac{1}{2}RDp$$
$$= \tfrac{1}{2}\times 120\times 72\times 3\tfrac{1}{2}$$
$$= 15{,}120 \text{ lb.}$$

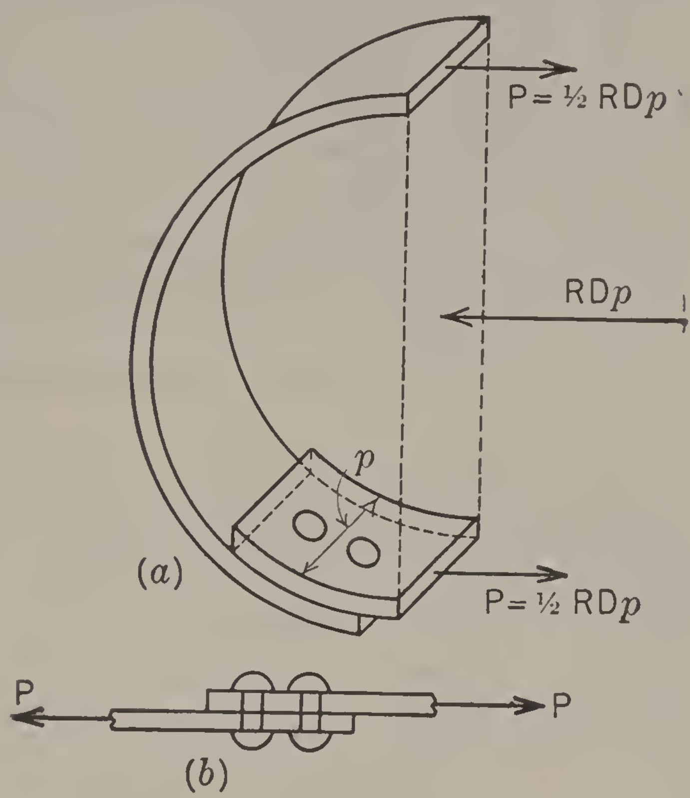


FIG. 34.—Stress in boiler joint.

The shearing unit-stress in the rivets then, is

$$s_s = \frac{P}{2\frac{\pi d^2}{4}} = \frac{15{,}120}{2\times 0.60} = 12{,}600 \text{ lb. per sq. in.}$$

The tensile unit-stress in the plate on a section through either row of rivets is

$$s_t = \frac{P}{(p-d)t} = \frac{15{,}120}{(3\frac{1}{2}-\frac{7}{8})\times\frac{7}{16}} = 13{,}200 \text{ lb. per sq. in.}$$

The bearing unit-stress of the plates against the rivets is

$$s_b = \frac{P}{2\cdot d\cdot t} = \frac{15{,}120}{2\times\frac{7}{8}\times\frac{7}{16}} = 19{,}800 \text{ lb. per sq. in.}$$

$$\text{Efficiency} = \frac{\text{least allowable strength of joint}}{\text{allowable strength of solid plate}}.$$

$$\text{Shearing efficiency } e_s = \frac{2\frac{\pi d^2}{4}\times 10{,}000}{p\cdot t\cdot 15{,}000} = \frac{\pi d^2}{2p\cdot t}\times\tfrac{2}{3} = 0.524 = 52.4 \text{ per cent.}$$

$$\text{Tensile efficiency } e_t = \frac{(p-d)t\ 15{,}000}{p\cdot t\cdot 15{,}000} = \frac{p-d}{p} = 0.75 = 75 \text{ per cent.}$$

$$\text{Bearing efficiency } e_b = \frac{2\cdot dt\cdot 22{,}500}{p\cdot t\cdot 15{,}000} = \frac{2d}{\text{p}}\times\frac{3}{2} = 0.75 = 75 \text{ per cent.}$$

Therefore, the efficiency of the joint is 52.4 per cent. If the joint were well proportioned the three efficiencies would be more nearly equal.

PROBLEMS

33. A boiler 30 in. in diameter is designed to withstand a pressure of 50 lb. per sq. in. The longitudinal joint is a single-riveted lap joint; the rivets have a pitch of 3 in.; the plates are $\frac{3}{8}$ in. thick and the rivets are $\frac{7}{8}$ in. in diameter. Find the shearing, tensile, and bearing unit-stresses in the joint. Also find the efficiency of the joint, using the allowable unit-tressses in stated Prob. 32 and Art. 14.

34. A column bracket (Fig. 35) consisting of a 6-in. by 6-in. by $\frac{3}{8}$-in. angle carries a load of 20,000 lb. and is riveted with five rivets $\frac{7}{8}$ in. in diameter to a 12-in., 30-lb. channel which forms part of a column. The thickness of the web of the channel is $\frac{1}{2}$ in. Find the shearing unit-stress in the rivets and the bearing unit-stress of the angle on the rivets.

Ans. $s_s = 6670$ lb. per sq. in., $s_b = 12{,}200$ lb. per sq. in.

35. A bridge post, Fig. 36, consists of two 10-in., 25-lb. channels latticed together. The total compressive stress P in the post is 275,000 lb. The post transmits this stress to a pin 6 in. in diameter by means of the bearing on the two channel webs and on two $\frac{5}{16}$-in. pin-

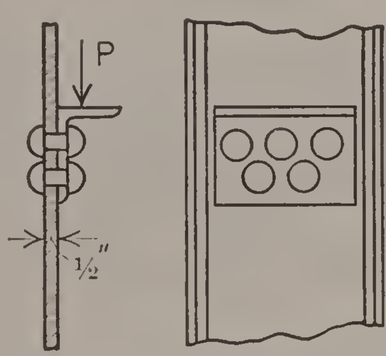


FIG. 35.—Column bracket.

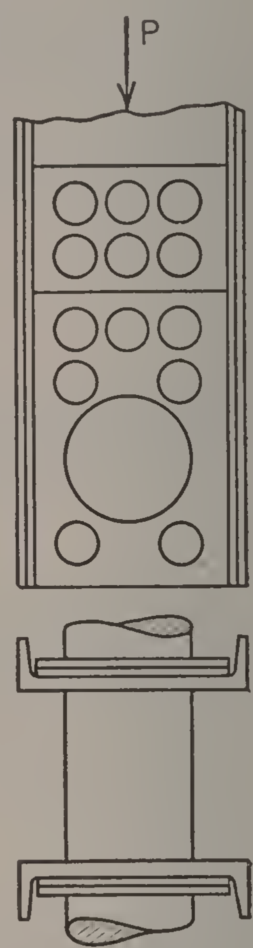


FIG. 36.—Column bearing on pin.

plates riveted to each channel, the thickness of each web being 0.53 in. The rivets are $\frac{7}{8}$ in. in diameter. Find the shearing unit-stress in the rivets and the bearing unit-stress on the pin.

36. The plates of a tank 60 in. in diameter are $\frac{3}{4}$ in. thick and are spliced by means of a double-riveted butt joint with two strap-plates. The strap-plates are $\frac{3}{8}$ in. thick. The rivets are staggered; the two lines are 3 in. apart, and the pitch on each line is $3\frac{1}{2}$ in. The rivets are $\frac{7}{8}$ in. in diameter. If the allowable unit-stresses are: $s_s = 10{,}000$ lb. per sq. in., $s_t = 15{,}000$ lb. per sq. in., and $s_b = 22{,}500$ lb. per sq. in., what is the maximum internal pressure to which the tank can be subjected. *Ans.* $R = 228$ lb. per sq. in.

37. A boiler 60 in. in diameter resists an internal pressure of 180 lb. per sq. in. The plates are $\frac{3}{4}$ in. thick and are spliced by means of a double-riveted butt joint with two strap-plates. The strap-plates are $\frac{3}{8}$ in. thick and the rivets are staggered; the two lines are 3 in. apart and the pitch on each line is also 3 in. The rivets are $\frac{7}{8}$ in. in diameter. Find the shearing, bearing and tensile unit-stresses.

Ans. $s_s = 6750$ lb. per sq. in., $s_b = 12{,}350$ lb. per sq. in., $s_t = 10{,}160$ lb. per sq. in.

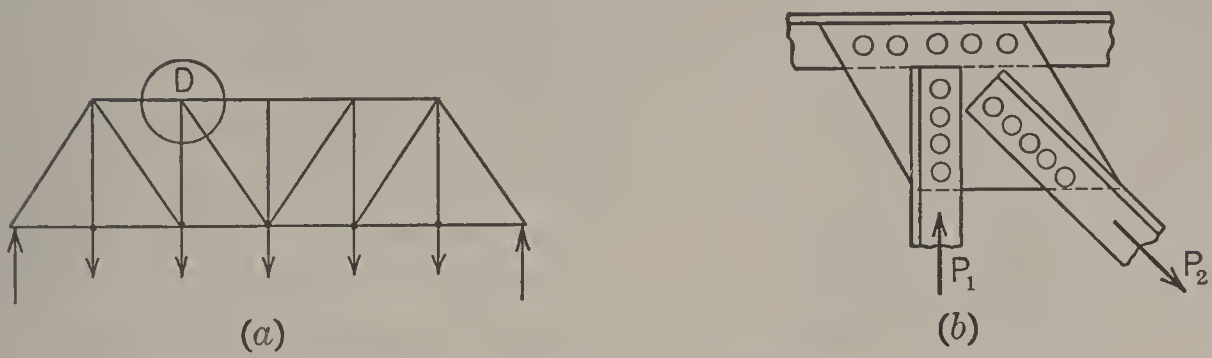


FIG. 37.—Riveted joint in truss.

38. In Fig. 37(*b*) is shown a joint that occurs in a riveted Pratt truss as at *D* in Fig. 37(*a*). If the rivets are $\frac{3}{4}$ in. in diameter and the allowable shearing unit-stress is 10,000 lb. per sq. in., what maximum values can P_1 and P_2 have? What is the bearing unit-stress on the rivets if the gusset plate and the angles are $\frac{3}{8}$ in. thick?

39. A triple-riveted lap joint is made of $\frac{9}{16}$ in. plates. The diameter of the rivets is 1 in.; the pitch in the outer rows is $4\frac{1}{2}$ in.; the pitch in the inner row is $2\frac{1}{4}$ in. and the distance between the rows of the rivets is $2\frac{1}{2}$ in. (*a*) What is the efficiency of this joint? (*b*) If this joint is the longitudinal seam of a 90-in. tank, what is the maximum allowable internal pressure? Use values for allowable unit-stresses stated in Prob. 32 and Art. 14.

Ans. $e_t = 77.8\%$, $R = 108$ lb. per sq. in.

40. A boiler 100 in. in diameter, having plates $\frac{3}{4}$ in. thick, is subjected to an internal steam pressure of 160 lb. per sq. in. The longitudinal joint is a double-riveted butt joint with two cover plates. The diameter of the rivets is $\frac{7}{8}$ in. and the pitch is 3 in. Find the bearing, shearing, and tensile unit-stresses in the joint.

13. Assumptions. Conditions Affecting Strength of Riveted Joints. *Friction.*—Since the rivets in a joint shrink while cooling, the plates are drawn tightly together which in turn causes friction between the plates when the joint is stressed. Tests [1] have shown, however, that slipping may occur at ordinary working stresses. Further, in any case, the amount of the friction is indeterminate, and hence the assistance obtained from the friction is usually neglected in the design of joints.

Bending and Tension.—In the design of a riveted joint, the assumption is usually made that only shearing stresses exist on a

[1] Bulletin 49, Engineering Experiment Station, University of Illinois.

cross-section of a rivet. The rivet, however, is always subjected to more or less bending, and the bending may have an important influence on the stresses in long rivets that connect several plates as indicated in Fig. 38. And, in a simple lap joint, bending also occurs as is indicated in Fig. 39. The effect of bending, however, is not disastrous if the material is ductile. Its influence in the design of the simpler types of joints is taken account of to some extent in selecting the values of the working unit-stresses.

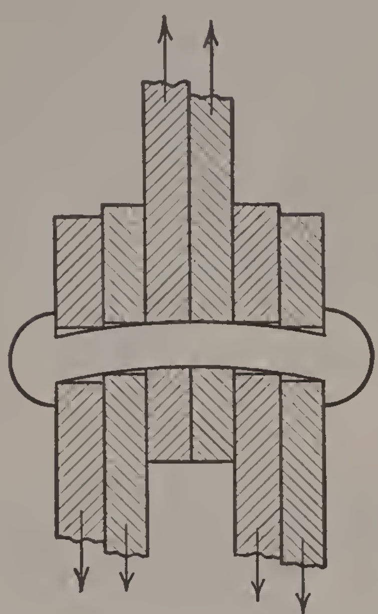


Fig. 38.—Bending in rivet.

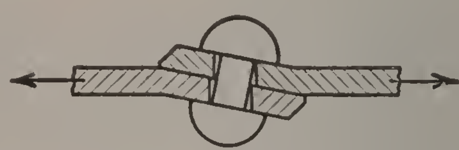


Fig. 39.—Bending in rivet.

Occasionally rivets must resist a direct tensile load as in the arrangement shown in Fig. 40. The allowable tensile stress in rivets should always be low and the thickness of the head of the rivet should be investigated to see if the shearing area in the head is ample; the contraction of the rivet during cooling causes stress in the rivet of unknown amount, but sufficient sometimes to cause the rivet head to snap off. Hence, rivets are not considered reliable for resisting direct tension.

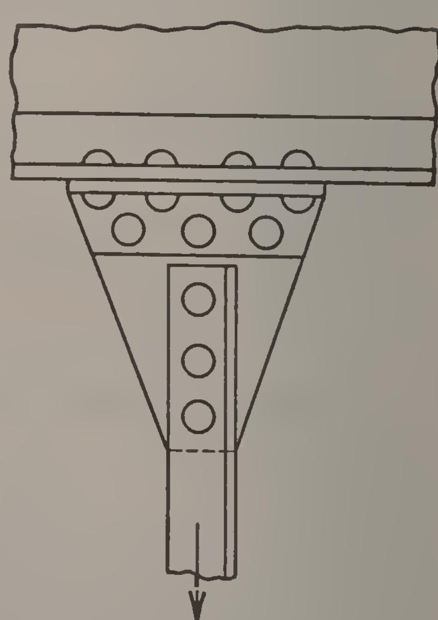


Fig. 40.—Rivets subjected to tensile stress.

Rivets are cheaper than bolts, otherwise bolts would be more generally used since they can be made safe in bending and tension as well as in shear. However, with the vibration that occurs in many structures it is difficult to keep the nuts of the bolts tight, and even the best nut-locks should be inspected frequently.

Rivet Holes.—Rivet holes are usually formed by punching the plates cold, for, although holes made by drilling are preferable, punching is the cheaper process. There are two objections to punching the hole: (1) The holes seldom come into exact alignment or register since only one plate can be punched at a time, and in order to cause the holes to register it is frequently necessary to ream out one or both holes or to use a drift pin. The latter method is particularly objectionable, but by either method the holes are enlarged and the rivets may not fill the holes, thereby causing excessive stress on some of the other rivets; (2) the tensile strength of the material around the hole is reduced due to the injury of the material accompanying the lateral flow of the metal caused by the punch.

Both of these difficulties may greatly be reduced by making the punched holes somewhat under-size and then reaming them to the correct size, after the plates or members are out together. Plates are seldom punched and used in their natural punched condition except in the case of thin plates and in the cheaper classes of work. The injury done to the plate by punching may be partially remedied by thorough annealing; it is generally impracticable, however, to anneal portions of large sheets merely for the sake of the rivet holes. Splice bars used in connecting railroad rails are frequently punched cold and then the whole bar annealed instead of being punched hot. In the best classes of marine work the law requires that all rivet holes shall be drilled from the solid plate, but the specifications for most structural work and for many pressure vessels require the holes to be punched small and reamed to the correct size. In obtaining the tensile unit-stress in a plate in which the holes are punched to size (not reamed) the diameter of the hole is sometimes assumed to be $\frac{1}{16}$ inch larger than that of the punched hole.

Methods of Riveting.—In structural work the riveting of some of the joints is done in the shop, whereas other joints must be riveted in the field during the erection of the structure. Shop riveting is usually done by machines which press the rivet in place and when well done is better than hand riveting. Since the rivet is a little smaller than the hole, the rivet while being driven, must be expanded throughout its whole length if it is to fill the hole, which is a necessary requirement for a good joint. Large rivets, therefore, almost always are machine riveted in order to obtain the heavy pressures required.

Specifications usually require that the allowable unit-stresses for hand-driven field rivets shall be less (often one-third less) than for machine-driven shop rivets since the conditions in the field are difficult to control. For example, the heating of the rivets varies; some rivets are burned, others are underheated and may be too cool when driven to be made to fill the hole, etc.

In designing joints the rivet is assumed to fill the hole and hence the stress, in compression members, is not increased due to having rivet holes in the member as is the case in tension members.

Distribution of Load Among Rivets.—If the load on a joint is a central load (passes through the centroid of the group of rivets) the assumption is made that the load is distributed equally to all the rivets (see Art. 89 for a discussion of eccentric loading). Obviously, it is impossible for the load to be distributed in this way, since the load on the rivets in one row depends on the yielding of the rivets in the row nearer to the load and also on the yielding of the plate between the rows, etc. The distribution of the load, therefore, is not easily determined, particularly in joints having several rows of rivets. But, in the absence of definite information the assumption stated above is used, particularly in the case of the simpler types of joints.

14. Allowable Stress.—The discussion, in the preceding articles, of the uncertainties in the actions occurring in joints should make it evident that the selection of the working or allowable unit-stresses in shear, tension, and bearing should be based not only on the strength of the materials of which the joint is made, but also on the results of tests of actual joints. Many tests on riveted joints have been made, and tests and experience indicate that in the design of joints of the usual proportions and with the methods of calculating stresses already discussed, the values of working stresses given below may be used. In using these values it is understood that the members or plates to be riveted meet the specifications for structural steel or boiler plate, etc., and that the rivets meet a similar requirement for rivet steel. Further, it is assumed that the rivet holes are punched small and reamed to size or are drilled from the solid plate, and that the rivets are shop driven. For joints having field-driven rivets or having holes punched to size (not reamed) the values given below should be reduced.

Shearing unit-stress in rivets;	$s_s = 10{,}000$ lb. per sq. in.
Tensile unit-stress in plate;	$s_t = 15{,}000$ lb. per sq. in.

Bearing unit-stress on rivets;	$s_b = 22{,}500$ lb. per sq. in.
Bearing unit-stress on pins;	$s_b = 22{,}500$ lb. per sq. in.
Compressive unit-stress in members;	$s_c = 15{,}000$ lb. per sq. in.

Thus the ratios of the shearing and bearing allowable unit-stresses to the tensile allowable unit-stress are as follows:

$$\frac{s_s}{s_s} = \frac{2}{3} \quad \text{and} \quad \frac{s_b}{s_t} = \frac{3}{2}.$$

These ratios are convenient for use in calculations in design.

PROBLEMS

41. In an ideal boiler or pipe joint the allowable strength (not allowable unit-stresses) in shear, bearing, and tensile are equal. (*a*) By equating the allowable shearing strength to the allowable bearing strength show that the diameter of the rivets in a *lap* joint is 2.87 times the diameter of the rivets ($d = 2.87t$). (*b*) Similarly, show that in an ideal butt joint having two cover plates, $d = 1.43t$. Which of these two types of joints is better suited for boilers, pipes, etc., having thick plates? Hint: If n represents the number of rivets we have for (*a*)

$$n \cdot \frac{\pi d^2}{4} = ntds_b \quad \text{or} \quad d = \frac{4}{\pi} \frac{s_b}{s_t} \cdot t.$$

42. By equating the allowable tensile strength to the allowable bearing strength show that the pitch in an ideal double-riveted lap joint is four times the diameter of the rivets ($p = 4d$). Also show that the efficiency of an ideal double-riveted lap joint is 75 per cent.

43. Find the relation between p and d, and also the efficiency, of an ideal double-riveted butt joint having two cover plates.

CHAPTER III[1]

ELEMENTARY COMBINED STRESSES AND COMBINED STRAINS. RESILIENCE

15. Introduction.—The intensities of stress that are most easily calculated directly from the loads on a member by use of the equations in the preceding chapters and those immediately following may not be the most significant stresses to which the member is subjected. Thus, if the intensity of stress at a point on one or more given planes in a body is known, it may be desired to determine the intensity of stress at the same point but on another plane passing through the point. Likewise, the relations between the strains in various directions may be desired. A more detailed discussion of these topics is given in Chapter XII.

16. Stresses on Oblique Section.—If an axial load P is applied

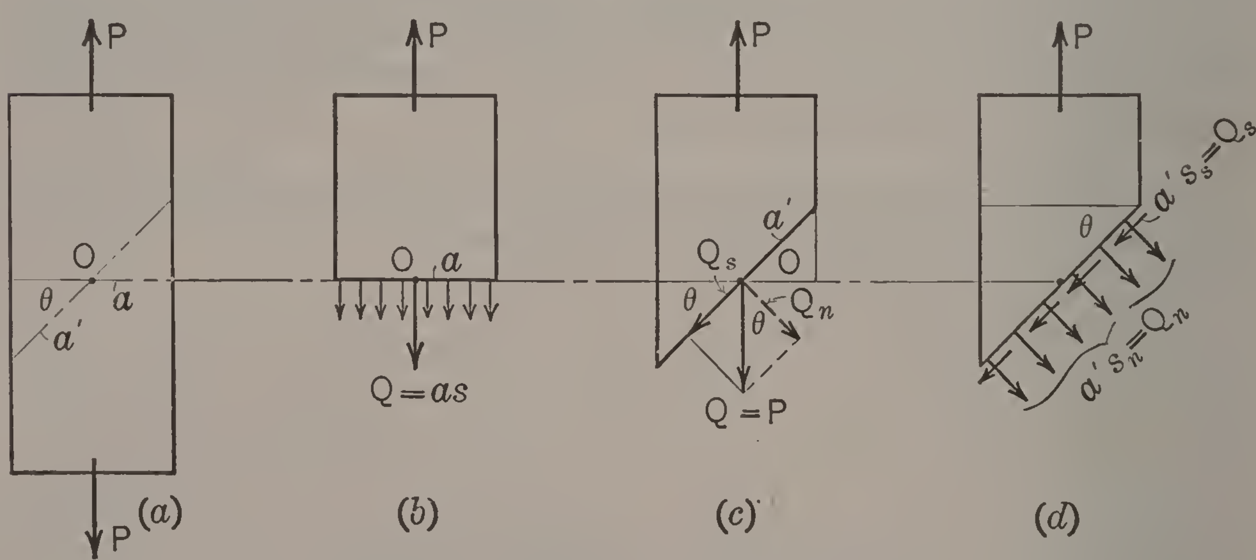


Fig. 41.—Stresses on oblique plane.

to a bar (Fig. 41a), the total stress, Q (Fig. 41b), on a cross-section perpendicular to the direction of the load is uniformly distributed on the area and hence is equal to the product of the area, a, of the

[1] This chapter may be omitted without causing difficulties for the student except, perhaps, in Chapter XII. It is strongly recommended, however, that Art. 16 be studied and as many more as time will permit.

section and the constant unit-stress, s. Further, since Q holds P in equilibrium it must be equal to, and collinear with, P and hence must pass through the centroid of the area a, that is, $P=Q=as$.

Now, if an oblique plane is passed through the centroid, O, of the area a and the lower part of the bar is removed (Fig. 41c), the total stress on the oblique area a' must likewise be Q in order to hold P in equilibrium. But Q is inclined to the area a' and, for convenience, will be resolved into normal (tensile) and tangential (shearing) components, Q_n and Q_s respectively (Fig. 41c), and since these components pass through the centroid of the area a' the normal and shearing intensities of stress, s_n and s_s on the areas a' are constant (Fig. 41d). Hence, the following equations may be written:

$$Q_n=a's_n=\frac{a}{\cos\theta}s_n, \quad \text{and} \quad Q_s=a's_s=\frac{a}{\cos\theta}s_s.$$

But

$$Q_n=Q\cos\theta=P\cos\theta \quad \text{and} \quad Q_s=Q\sin\theta=P\sin\theta.$$

Therefore,

$$P\cos\theta=\frac{a}{\cos\theta}s_n \quad \text{and} \quad P\sin\theta=\frac{a}{\cos\theta}s_s$$

or

$$s_n=\frac{P}{a}\cos^2\theta \quad \text{and} \quad s_s=\frac{P}{a}\sin\theta\cos\theta=\frac{1}{2}\frac{P}{a}\sin 2\theta, \qquad (16)$$

from which the normal and shearing unit-stresses on any section inclined at an angle θ to the section on which the maximum normal stress occurs may be found. Now s_n will be a maximum when θ is zero, that is, on the plane perpendicular to P, and its value is $s=\dfrac{P}{a}$ which agrees with the value found in Art. 3. The maximum value of s_s in the above equation, however, occurs when θ is 45° and hence,

$$\text{maximum value of } s_s=\frac{1}{2}\frac{P}{a}.$$

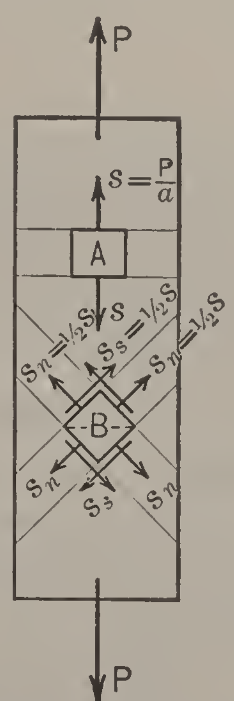


FIG. 42.—Stresses on 45-degree planes in tension member.

Thus the maximum value of the shearing unit-stress in a bar subjected to an axial tensile (or compressive) load is equal to one-half the maximum normal stress developed in the bar, and it occurs on planes making angles of 45° with the plane on which the maximum normal stress occurs. But, since the shearing strength of many materials is much less than the tensile or compressive strength, the shearing unit-stress developed may be the more significant stress. The stresses acting on the faces of a small block in the bar, the faces of the block being perpendicular and parallel to the direction of P, are shown at A in Fig. 42, and the stresses on a block having faces making angles of 45 degrees with the direction of P are shown at B in Fig. 42.

PROBLEMS

44. A brick 2 in. by 4 in. by 8 in. is tested in compression by applying an axial load on its 2-in. by 4-in. ends. If the ultimate compressive and shearing strengths are 3000 lb. per sq. in. and 1000 lb. per sq. in., respectively, will the brick fail when subjected to a load of 20,000 lb.?

45. A structural steel bar $\frac{3}{4}$ in. in diameter and 10 in. long is subjected to an axial tensile load of 12,000 lb. (*a*) Find the tensile and shearing unit-stresses on a plane making 60° with the direction of the load. (*b*) Find the maximum shearing unit-stress.

Ans. (*a*) $s_n = 20{,}300$ lb. per sq. in.; $s_s = 11{,}700$ lb. per sq. in. (*b*) $s_s = 13{,}600$ lb. per sq. in.

46. Show that when a bar is subjected to an axial load the normal unit-stress, s_n, on the section having the maximum shearing unit-stress is $\frac{1}{2}\frac{P}{a}$, and hence has the same value as the maximum shearing unit-stress (see Fig. 42).

47. Show that the forces (stresses) s_s and s_n acting on the faces (unit-areas) of the cube B in Fig. 42 cause an intensity of stress on the horizontal diagonal plane through the cube equal to s or $\frac{P}{a}$.

17. Shearing Stresses on Planes at Right Angles. Simple Shear. *Proposition.*—If a shearing stress of intensity s_s occurs on a plane at a given point in a body there must exist a shearing stress of equal intensity at that point on a plane at right angles to the first plane.

Proof.—Let an elementary rectangular block (Fig. 43*a*) be removed from a body in which the block is subjected to shearing stresses on a pair of parallel faces, such as faces $\overline{AB}$ and $\overline{CD}$ or faces $\overline{AC}$ and $\overline{BD}$. For example, the block may be in a bolt as shown in Fig. 43(*b*), where the part X causes a shearing stress of average intensity s_h on the horizontal plane $\overline{AB}$ of part Y, and the part Z exerts a resisting stress of equal intensity on the face $\overline{CD}$. Similarly, the part X of the shaft in Fig. 43(c) causes an average shearing unit-stress s_v on the vertical plane $\overline{BD}$ and the part Z of the shaft exerts an equal resisting or opposite stress on the face $\overline{AC}$.

Now the shearing forces acting on a pair of faces form a couple and since the block is in equilibrium when in the body, there must be shearing forces on the other pair of faces such that the moments of the two couples are equal. If the depth of the block perpendicular to the paper is assumed to be unity, then

$$\overline{AB}s_h \cdot \overline{BD} = \overline{BD}s_v \cdot \overline{AB}$$

whence

$$s_h = s_v. \qquad (18)$$

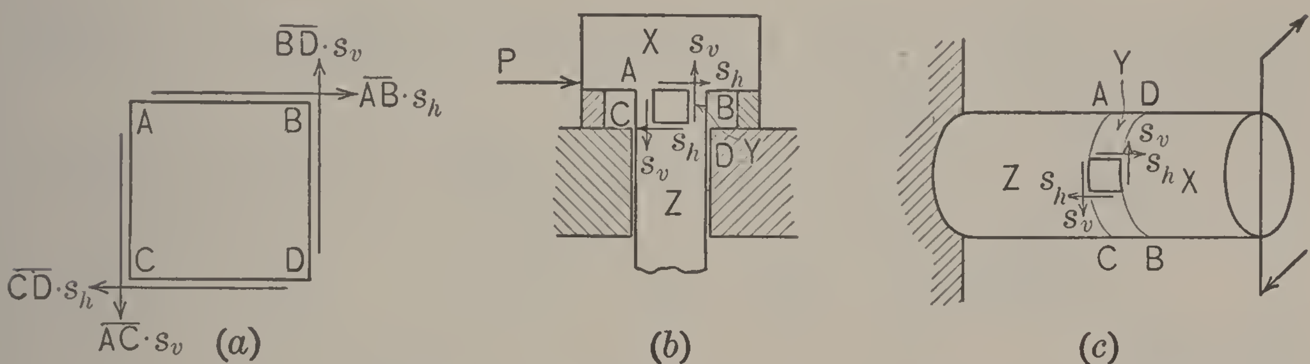


FIG. 43.—Shearing stresses of equal intensity on planes at right angles to each other.

Now if the dimensions of the block are considered to be indefinitely small, s_h and s_v may be considered to be the stress at a point. That is, the intensities of the shearing stresses on planes at right angles to each other at any point in a body are equal; and this is true also when normal stresses act on the planes in addition to shearing stresses, as is illustrated in Fig. 42. If shearing stresses, *only*, occur on the two planes at any point in the body, the body at that point is said to be in a state of *simple shear* or of *pure shear*.

18. Tensile and Compressive Stresses Resulting from Simple Shear. *Proposition.*—If a state of simple shear exists at a point in a body (as in Fig. 43c) there also exists normal (tensile and compressive) stresses on planes that bisect the planes on which the shearing stresses occur, and the intensities of the normal stresses are equal to those of the shearing stresses.

Proof.—Let a diagonal plane $\overline{AD}$ be passed through the block of Fig. 44(a), or Fig. 43(a); the forces acting on one part of the block are shown in Fig. 44(*b*). The face of the block in Fig. 44 is assumed, for convenience, to have a thickness (perpendicular to the plane of the paper) of unity. The force on each face is equal to the area of the face times the unit-stress on that face, and since

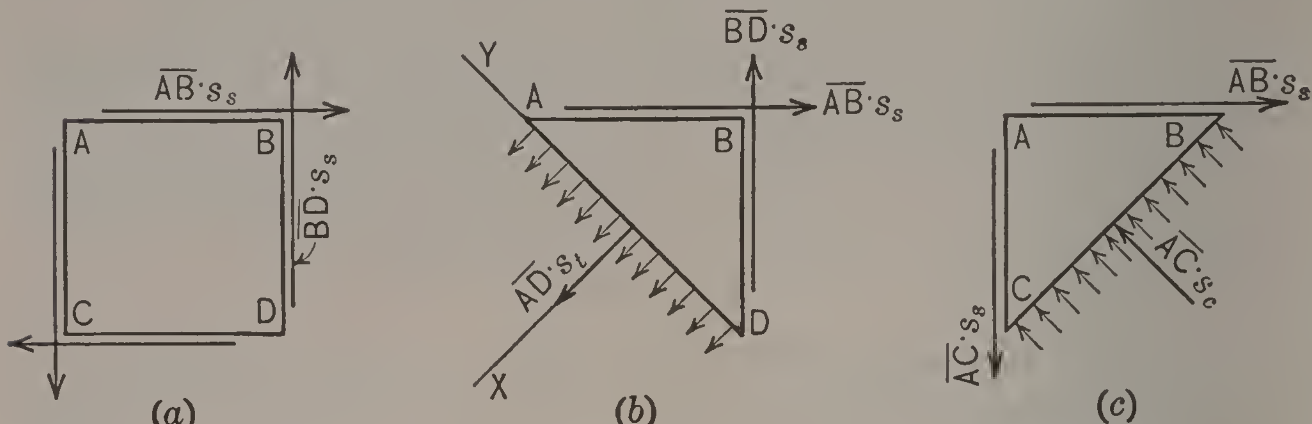


FIG. 44.—Tensile and compressive stresses resulting from simple shear.

the block ABD (Fig. 44*b*) is in equilibrium, we have, by resolving the forces in the x-direction,

$$\overline{AD}s_t = \overline{AB}s_s \cos 45° + \overline{BD}s_s \cos 45° = s_s(\overline{AB} \cos 45° + \overline{BD} \cos 45°)$$

but

$$\overline{AB} \cos 45° + \overline{BD} \cos 45° = \overline{AD}.$$

Therefore,

$$s_t = s_s.$$

Similarly, from Fig. 44(*c*),

$$s_c = s_s. \qquad \qquad (19)$$

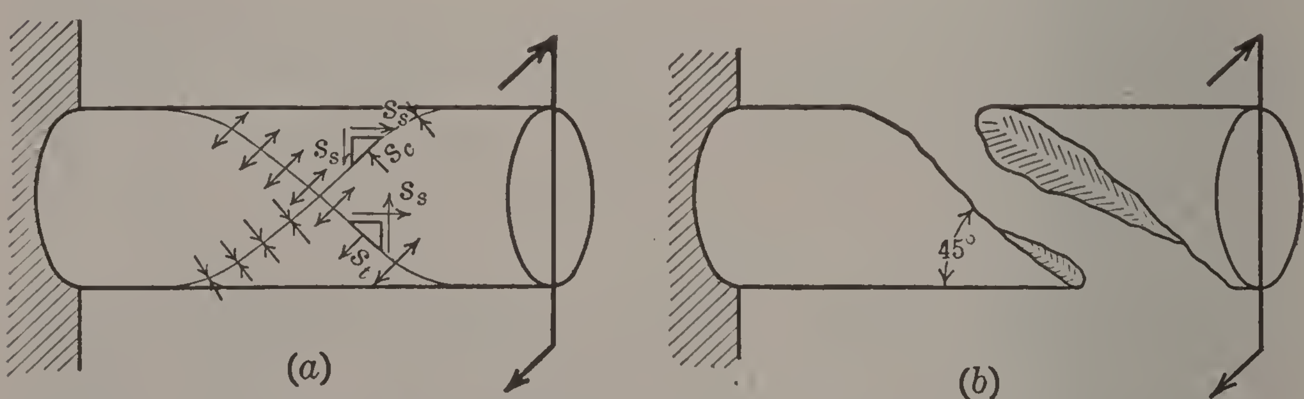


FIG. 45.—Brittle material when subjected to torsion fails in tension.

Thus when a brittle material such as cast iron, which is weak in tension, is twisted as in Fig. 43(*c*) and Fig. 45(*a*) the material fails in tension on a plane inclined (approximately 45°) to the planes on which the shearing stresses occur, as indicated in Fig. 45(*b*).[2]

[2] The student may perform an experiment on a crayon of chalk to show the action indicated in Fig. 45.

19. Principal Stresses.—With any combination of stresses at a point in a body three planes can be found passing through the point, on which only normal stresses exist; the normal stresses on these planes, on which no shearing stresses occur, are called *principal* stresses. In many problems in the following chapters two of the three principal stresses are equal to zero; in some problems only one of the principal stresses is equal to zero.

Maximum normal stresses are always principal stresses, and hence principal stresses are of much importance in engineering problems; they will be discussed in greater detail in Chapter XII.

20. Shearing Stresses Resulting from Principal Stresses. *Proposition.*—If normal (principal) stresses, only, occur on three planes at right angles to each other at a point in a body, shearing

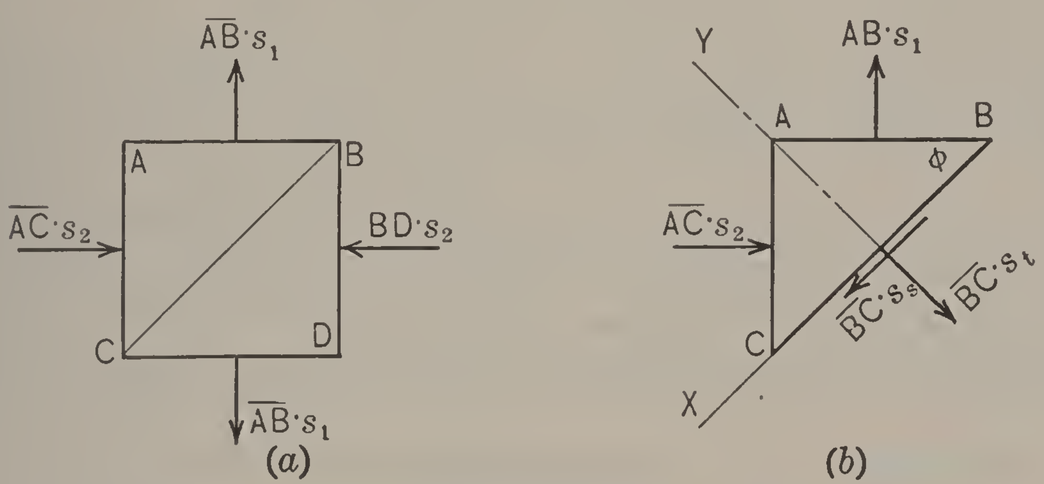


FIG. 46.—Shearing stress resulting from principal stresses.

stresses occur on oblique planes through this point; the maximum value of the shearing unit-stress is one-half of the algebraic difference of the maximum and minimum principal unit-stresses, that is,

$$\text{Max. } s_s = \tfrac{1}{2}(s_{\text{max.}} - s_{\text{min.}})$$

in which a compressive stress is to be considered a negative tensile stress. Further, the maximum shearing unit-stress occurs on each of the two planes that bisect (make 45° with) the planes on which the maximum and minimum normal stresses occur.

Proof.—Let the block in Fig. 46(*a*) represent a small part of a body subjected to a tensile principal unit-stress s_1 on the plane AB (and CD), a compressive principal stress s_2 on the plane BD (and AC), and a principal stress of zero magnitude on the faces parallel to the plane of the paper (bodies subjected to this state of stress will be considered in Chap. XII.); the maximum principal unit-stress is $+s_1$ and the minimum is $-s_2$.

Now if an oblique plane is passed through the block and a part of the block is removed, the forces holding the remaining part in equilibrium are shown in Fig. 46(*b*). By applying one of the equations of equilibrium (resolving the forces parallel to the x-axis) the following equation is found;

$$\overline{BC}\, s_s = \overline{AB}\, s_1 \sin\phi + \overline{AC}\, s_2 \cos\phi.$$

Hence

$$s_s = [s_1 - (-s_2)] \sin\phi \cos\phi$$

$$= \tfrac{1}{2}(s_{\text{max.}} - s_{\text{min.}}) \sin 2\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

But $\sin 2\phi$ is a maximum (equal to unity) when ϕ equals $45°$, and hence the above proposition is shown to be true.

It should be noted that if s_1 and s_2 are of like sign (both tensile or both compressive stresses) and the third principal stress is zero, as in the shell of a boiler (Fig. 48*b*), then the minimum stress is zero, and hence the maximum shearing unit-stress is $\frac{1}{2}s_{\text{max}}$. And, if two principal stresses are zero (as in Fig. 41) the maximum shearing unit-stress is merely $\frac{1}{2}s$ where s is the only principal stress, which agrees with the value of s_s found in Art. 16.

PROBLEMS

48. A boiler 6 ft. in diameter is made of plates $\frac{3}{4}$ in. thick and is subjected to an internal steam pressure of 200 lb. per sq. in. Find the maximum shearing unit-stress in the plate. *Ans.* $s_s = 4800$ lb. per sq. in.

49. A gun barrel or thick cylinder (Fig. 47) on which hoops are shrunk (not shown in Fig. 47) is subjected to an internal pressure of 50,000 lb. per sq. in. due to the explosion of the charge; the maximum radial compressive stress, s_c, in the material is also 50,000 lb. per sq. in. Now if the maximum circumferential stress, s_t, is 16,000 lb. per sq. in. and occurs at the same point in the cylinder as does s_c, find the maximum shearing unit-stress and indicate the planes on which it occurs.

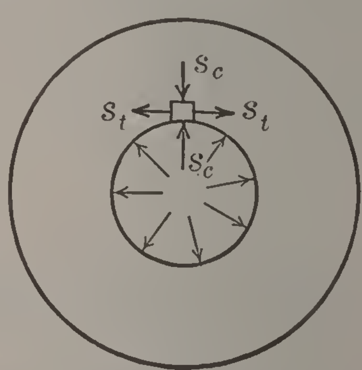


Fig. 47.—Stresses in thick cylinder.

50. At a certain point in a material subjected to stress a compressive stress of 2000 lb. per sq. in. exists in a direction at right angles to a tensile stress of 2000 lb. per sq. in. Find the normal unit-stress and the shearing unit-stress on a plane making 30° with the direction of the tensile stress.

Ans. $s_t = 5320$ lb. per sq. in. $s_s = 14{,}300$ lb. per sq. in.

21. Poisson's Ratio and the Relation between Moduli of Elasticity.—If a bar is subjected to an axial tensile load the bar is elongated in the direction of the load, that is, in the longitudinal direction, and at the same time the lateral dimension of the bar decreases. The ratio of the lateral unit-strain to the longitudinal unit-strain is called *Poisson's ratio* and will be denoted by the symbol m. The value [3] for this ratio for steel (and most structural metals) is approximately $\frac{1}{4}$; values of $\frac{1}{4}$ to $\frac{1}{3}$ are frequently used.

The shearing modulus of elasticity, E_s, of a homogeneous material may be expressed in terms of the tensile (or compressive) modulus of elasticity, E, and Poisson's ratio, m, by the following equation.[4]

$$E_s=\frac{E}{2(1+m)}. \qquad (22)$$

And since both E_s and E may be obtained from experimental data (see Art. 5, 144 and 145), this equation offers a convenient means of determining Poisson's ratio.

If the value of m for steel is taken to be $\frac{1}{4}$ then the above expression (for steel) becomes $E_s=\frac{2}{5}E$. (Compare the ratio of the moduli of elasticity as given by this equation with the experimental values in Table 1).

22. Strains due to Principal Stresses.—If a rectangular block

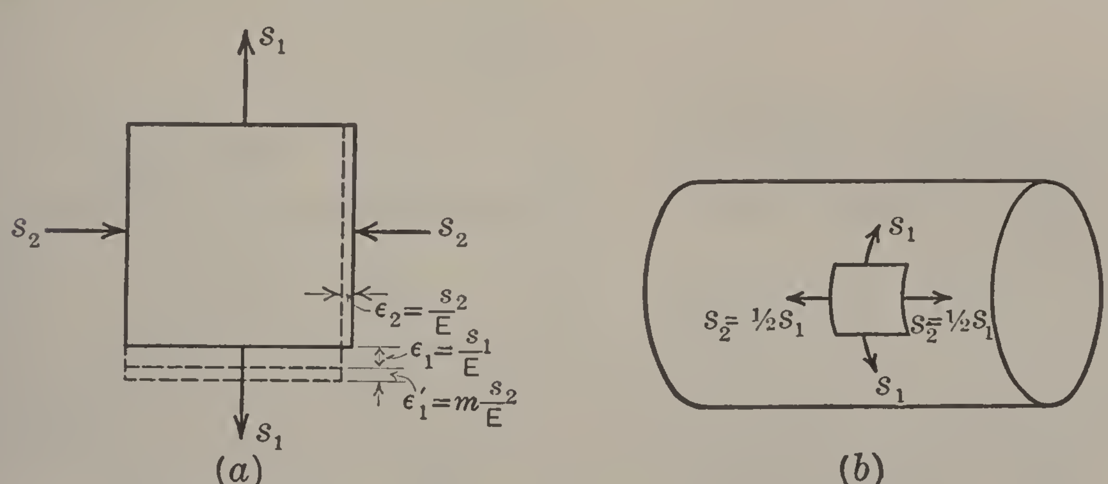


FIG. 48.—Strain due to two principal stresses.

of material is subjected to normal stresses, only (that is, to principal stresses), on two pairs of faces (Fig. 48a)—tensile unit-stresses s_1

[3] See "Determination of Poisson's Ratio" by T. M. Jasper, in Proceedings of Am. Soc. for Testing Materials, 1924.

[4] The derivation of this equation is beyond the scope of this book; it may be found in treatises on the theory of elasticity.

on one pair and compressive unit-stresses s_2 on the other pair—the unit-elongation in the direction of s_1 caused by s_1 is

$$\epsilon_1 = \frac{s_1}{E},$$

and the unit-strain, ϵ_2, in the direction of s_2 caused by s_2 is

$$\epsilon_2 = \frac{s_2}{E},$$

and the unit-strain ϵ_1' in the direction of s_1 caused by s_2 is

$$\epsilon_1' = m\,\epsilon_2 = m\frac{s_2}{E},$$

in which the modulus of elasticity, E, is assumed to be the same in compression as in tension. Therefore, the total unit-strain in the direction of s_1 is

$$\epsilon = \frac{s_1}{E} + m\frac{s_2}{E},$$

or

$$E\epsilon = s_1 + ms_2. \qquad (23)$$

And if both stresses are tensile stresses as occurs in the shell of a boiler (Fig. 48*b*), and in other structures, the equation becomes

$$E\epsilon = s_1 - ms_2. \qquad (24)$$

It is important to note that unit-stress is proportional to unit-strain $\left(\frac{s}{\epsilon} = E\right)$ only when the material is subjected to one principal stress. Or, to put the same idea in other words, the normal unit-stress in a material is equal to the modulus of elasticity times the unit-strain in the direction of the stress ($s = E\epsilon$) *only* when the material is subjected to a normal stress in one direction. The use and significance of the term $E\epsilon$ is discussed in Art. 112 and 113.

23. Resilience.—Resilience is the property of a material that enables it to give up energy (do work) when the stress in it is released. The amount of energy released or recovered per unit-volume of the material when the intensity of stress decreases from the proportional limit to zero is called the *modulus of resilience.* Now the energy recovered when the stress is released from the proportional limit may be assumed to be equal to the work done on the material in stressing it to the proportional limit; that is, within the proportional limit, the energy lost in stressing the material may be assumed to be negligible.[5] The property of resilience is of special importance in connection with impact and energy loads and is discussed further in Chapter XIII and also in Art. 146.

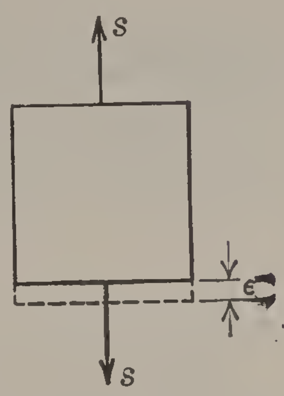


FIG. 49.—Block subjected to one principal stress.

One Normal Stress.—Let a unit-volume (cube) of material be subjected to a normal unit-stress s in one direction only, as indicated in Fig. 49. If the stress increases gradually from zero value and causes the cube to elongate an amount ϵ, the work done, w, is

$$w=\tfrac{1}{2}s\epsilon, \quad . \quad . \quad . \quad . \quad (25)$$

and since within the proportional limit $\epsilon=\frac{s}{E}$, the work done per unit-volume in stressing the material to any value s less than the proportional limit is

$$w=\frac{1}{2}\frac{s^2}{E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and hence the tensile (or compressive) modulus of resilience, k, is

$$k=\frac{1}{2}\frac{s_e^2}{E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

in which s_e is the proportional limit of the material.

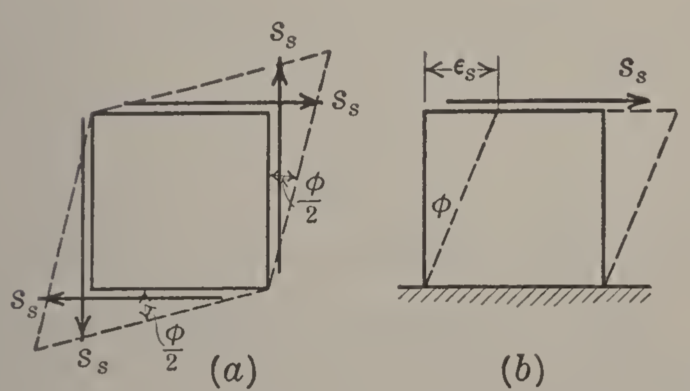


FIG. 50.—Block subjected to pure shear.

Pure Shear.—Let a unit-volume of a material be subjected to pure shear as indicated in Fig. 50(*a*) or as indicated more conveniently in Fig. 50(*b*). The work done in gradually increasing the stress from zero to the value s_s as the shearing strain increases to ϵ_s is

$$w_s=\tfrac{1}{2}s_s\epsilon_s,$$

[5] If the stress is repeated a great many times the total energy lost may not be negligible although the loss of energy in any one cycle of stress may be negligible.

and if s_s does not exceed the shearing proportional limit $\epsilon_s=\frac{s_s}{E_s}$, and hence the shearing modulus of resilience, k_s, is

$$k_s=\frac{1}{2}\frac{(s_s)_e^2}{E_s}, \quad \ldots \ldots \ldots \ldots \quad (29)$$

in which $(s_s)_e$ is the shearing proportional limit.

Combined Shearing and Normal Stresses.—If a cube of material is stressed as indicated in Fig. 51 the work done per unit-volume is

$$w=\frac{1}{2}\frac{s^2}{E}+\frac{1}{2}\frac{s_s^2}{E_s}. \quad \ldots \ldots \ldots \ldots \quad (30)$$

This combination of stresses will be discussed in Chapter XII, and the above expression is used in Art. 113.

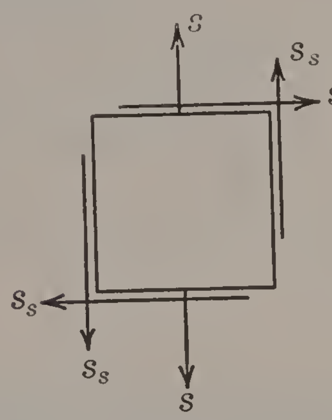


FIG. 51.—Block subjected to tension and shear.

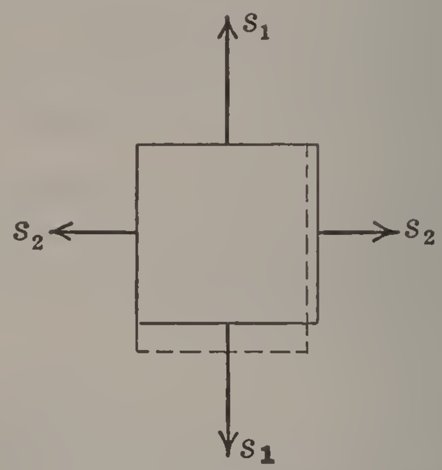


FIG. 52.—Block subjected to two principal stresses.

Two Principal Stresses.—If a unit-volume of material is subjected to normal stresses, only, on two faces (that is, to two principal stresses) as shown in Fig. 52, the work done may be found as follows: If s_1 were acting alone the work done would be $\frac{1}{2}s_1\epsilon_1$, where ϵ_1 is the deformation due to s_1. Now if s_2 be applied it does an amount of work equal to $\frac{1}{2}s_2\epsilon_2$, where ϵ_2 is the strain in the direction of s_2 due to s_2. But the strain ϵ_2 causes a strain equal to $m\epsilon_2$ in the direction opposite that of s_1, and hence s_1 does an additional amount of work equal to $-s_1\cdot m\epsilon_2$. Therefore the total work done is

$$w=\tfrac{1}{2}s_1\epsilon_1+\tfrac{1}{2}s_2\epsilon_2-s_1m\epsilon_2$$

$$=\frac{1}{2}\frac{s_1^2}{E}+\frac{1}{2}\frac{s_2^2}{E}-m\frac{s_1s_2}{E}, \quad \ldots \ldots \ldots \quad (31)$$

in which a tensile stress is to be taken as positive and a compressive strength as negative.

PROBLEMS

51. Calculate the tensile modulus of resilience of structural steel using the values in Table 1. *Ans.* $k=20.4$ in.-lb. per cu. in.

52. If the amount of energy that can be stored in a material before inelastic action begins is a constant regardless of the way the material is stressed, show that the shearing proportional limit of steel is 0.63 times the tensile proportional limit, assuming $E_s=\frac{2}{5}E$.

CHAPTER IV

STRESS AND DEFORMATION DUE TO TORSIONAL LOADS

24. Torsional Load, Twisting Moment and Resisting Moment Defined. *Torsional Load.*—Forces that cause a bar to twist about its central axis are called *torsional loads.* The resultant of torsional loads acting on a shaft is a twisting couple (two equal, opposite, non-collinear forces) since simple turning or twisting can be produced only by a couple. The resultant moment (or moment of the resultant couple) which causes the twisting is the algebraic sum of the moments of the torsional loads about the axis of the bar or shaft.

Twisting Moment.—The *twisting moment* for a section of a shaft is the name given to the algebraic sum of the moments, about the axis of the shaft, of the torsional loads that lie to one side of the section. The symbol T will be used to denote the twisting moment.

It is here assumed that the torsional loads lie in planes perpendicular to the axis of the shaft. If the forces do not lie in such planes the forces may be resolved into components perpendicular and parallel, respectively, to the axis, the components parallel to the axis being neglected in the following discussion.

Resisting Moment.—In Fig. 53(b) is shown a shaft subjected to a twisting moment T (equal to Pp) at one end of the shaft. Now since the shaft is in equilibrium the body A must exert an equal and opposite moment (not shown) on the other end of the shaft and these two moments cause the shaft to twist.

The shaft when thus twisted has shearing stresses developed on each normal section of the shaft and the moment of these shearing stresses on any section, called the resisting moment, holds the twisting moment in equilibrium. For example, let a plane, BC, be passed through the shaft and let the portion of the shaft to the right of the plane be removed. Now if the remaining (left) portion of the shaft (Fig. 52*a*) is held in equilibrium (as it was when in the whole shaft) a moment equal and opposite to T must be applied to it. This resisting moment (denoted by T_r, Fig. 52*b*) is exerted on the left portion by the right portion on the section DE cut by the plane, and is due to the shearing stresses developed on the section. Thus, the shearing *resisting moment* at a section of a shaft is the sum of the moments, about the center of the shaft, of the shearing stresses that are developed on the section in order to

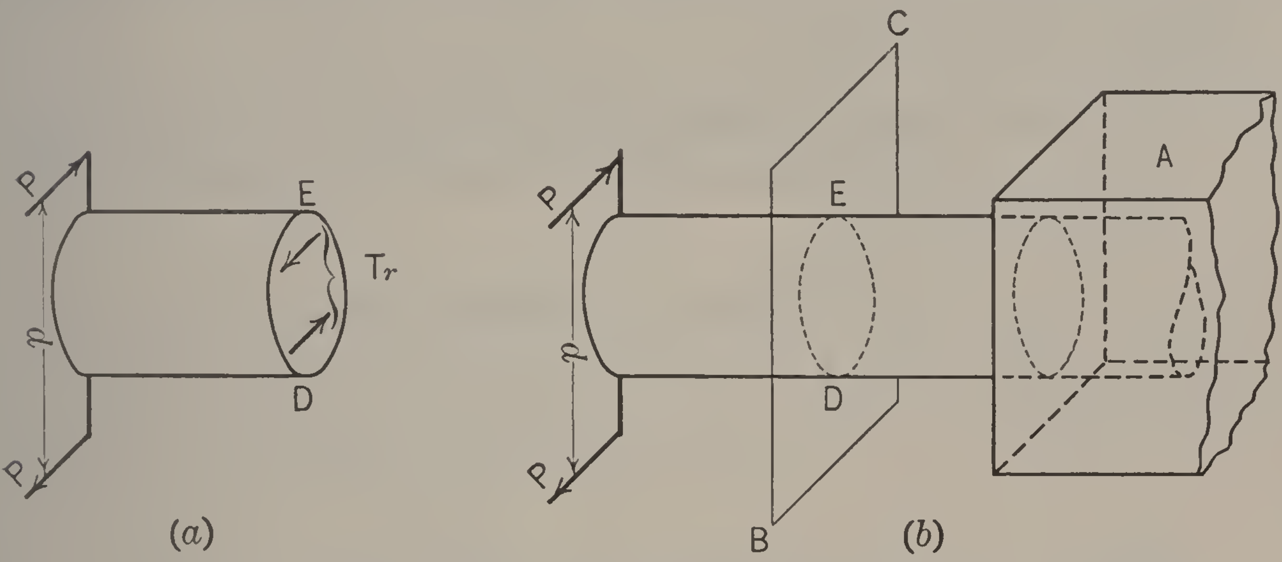


FIG. 53.—Cylindrical bar subjected to a twisting moment.

hold the twisting moment for this section in equilibrium. Thus from the condition of equilibrium, we have,

Moment of External forces = Moment of internal forces or stresses, that is,

$$\text{Twisting moment} = \text{Resisting moment}$$

or

$$T = T_r.$$

25. Distribution of Stress on the Cross-section.—Now before an expression for the sum of the moments of the stresses on the area (resisting moment) can be found, the way in which the stress is distributed over the area must be determined. It is assumed that the intensity of shearing stress (shearing unit-stress) is zero at the center of the section and increases directly as the distance from the central axis.

In order to supply the evidence or reasoning leading to this assumption it is necessary to consider:

(*a*) the manner in which the shearing strains of the fibers in the shaft vary when the shaft is twisted.

(*b*) the relation between the shearing strain of a material and the accompanying shearing stress in the material.

Information concerning these points is obtained mainly from the results of experiments:

(*a*) Experiments have shown that when a cylindrical[1] shaft is twisted, a plane section of the shaft before twisting is approximately a plane section after twisting, and it is assumed that a diameter in the section before twisting is a diameter (straight line) after twisting.[2] If this is true,

> The shearing *unit-strain* of the material at any point in the shaft is proportional to the distance of the point from the central axis.

Proof.—Fig. 54(*a*) represents a cylindrical shaft twisted by the twisting moment T or Pp. An outer element or fiber AB, having

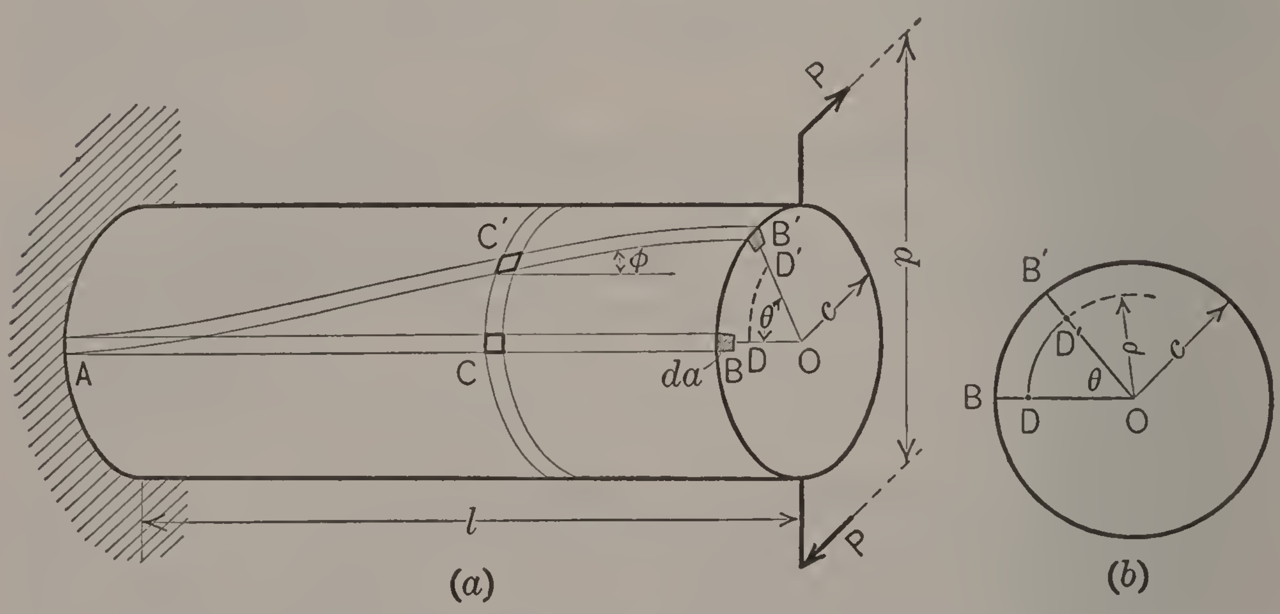


Fig. 54.—Shearing strain in shaft due to twisting moment.

a cross-sectional area da, takes the form of a helix AB' when the shaft is twisted. Now a small portion of the fiber, which is at C before the shaft is twisted is at C' after the shaft has been twisted

[1] A plane section of a shaft whose cross-section is not circular does not remain plane when the shaft is twisted. Torsion of shafts having non-circular sections is discussed very briefly in Art. 31.

[2] Although this assumption seems a reasonable one, it is very difficult to obtain direct experimental verification. The justification for the assumption is to be found in the agreement of results calculated from the formula based on this assumption and experimental results.

and is deformed as shown in the figure by the shearing stresses on its faces.

Further, as discussed in Art. 4, the shearing unit-strain, ϵ_s, at any point of a body is equal to the tangent of the angle of deviation from a right angle of two planes passing through the point. Therefore, the shearing unit-strain, ϵ_s, at the surface of a cylindrical shaft when twisted is

$$\epsilon_s = \tan \phi.$$

But $\tan \phi$ (Fig. 54*a*) is equal to the arc $\overline{BB'}$ divided by the length, l, of the shaft, and further, $\overline{BB'}$ is equal to the radius c times the angle of twist, θ, expressed in radians. Therefore,

$$\epsilon_s, \text{ at surface,} = \tan \phi = \frac{BB'}{l} = \frac{c\theta}{l}.$$

Similarly, the shearing unit-strain at a point at the distance ρ from the center of the shaft (Fig. 54*b*) is

$$\epsilon_s, \text{ at distance } \rho, = \frac{DD'}{l} = \frac{\rho\theta}{l}.$$

Now since a radius OB before the shaft is twisted is a straight line, OB', after twisting, and since the lengths, l, of all fibers are equal, it follows that

$$\frac{\epsilon_s \text{ at distance } c \text{ from center}}{\epsilon_s \text{ at distance } \rho \text{ from center}} = \frac{\frac{c\theta}{l}}{\frac{\rho\theta}{l}} = \frac{c}{\rho}.$$

That is, when a shaft is twisted the shearing unit-strains of the fibers vary directly as the distances of the fibers from the central axis of the shaft.

(*b*) As stated in Art. 5, when a material is stressed (whether in tension, compression or shear), experiments have shown that for nearly all structural materials the unit-stress at any point in the material is proportional to the corresponding unit-strain at that point provided that the unit-stress does not exceed the proportional limit of the material. Hence if $(s_s)_c$ and $(\epsilon_s)_c$ denote the shearing unit-stress and unit-strain, respectively, of fibers at the distances c from the axis, and the subscript ρ is used in a similar way, then

$$\frac{(s_s)_c}{(\epsilon_s)_c}=\frac{(s_s)_\rho}{(\epsilon_s)_\rho}=\text{etc.}=\text{a constant.}$$

The numerical value of this constant ratio is called the shearing modulus of elasticity of the material and is denoted by E_s (Art. 5).

Distribution of Stress over Area.—The way in which the shearing unit-stress varies over the area may now be stated. For, since the shearing unit-strains of the fibers are proportional to the distances of the fibers from the center of the shaft, and since the shearing unit-strains of the fibers are also proportional to the shearing unit-stresses on the fibers it follows that:

> The shearing unit-stress on a fiber of the shaft varies directly as the distance of the fiber from the axis of shaft (as shown in Fig. 55). Or, expressed mathematically,

$$\frac{(s_s)_c}{(s_s)_\rho}=\frac{c}{\rho} \quad \text{or} \quad \frac{(s_s)_c}{c}=\frac{(s_s)_\rho}{\rho}=\text{etc.}=\text{a constant} \quad . \quad . \quad (32)$$

26. Expression for Resisting Moment. The Torsion Formula.—The sum of the moments of the shearing stresses (resisting moment) may now be found as follows:

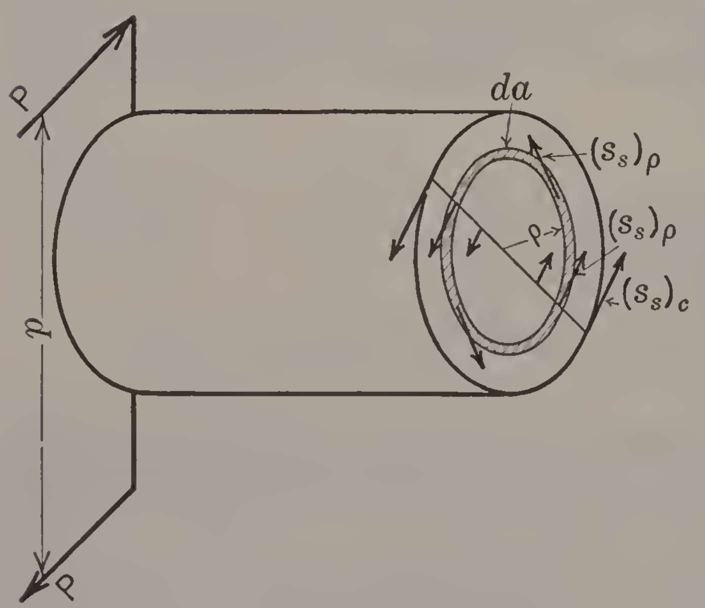


FIG. 55.—Shearing stresses in cylindrical shaft.

The unit-stress at the distance ρ from the axis (Fig. 55) is $(s_s)_\rho$ and the unit-stress may be assumed to be constant over the element of area da. Hence,

Total shearing stress on area $da=(s_s)_\rho\, da$

Moment of total stress on area $da=(s_s)_\rho da\cdot\rho$

Sum of moments of stresses on all elements of area, about axis of shaft $=T_r=\int (s_s)_\rho\, da\rho$

This may be written

$$T_r = \int \frac{(s_s)_\rho}{\rho} \, da\rho^2,$$

but, as shown above, $\dfrac{(s_s)_\rho}{\rho}$ is constant and equal to $\dfrac{(s_s)_c}{c}$, and hence the above expression may be written

$$T_r = \frac{(s_s)_c}{c} \int da\rho^2.$$

Now $\int da\rho^2$ is the polar moment of inertia of the area with respect to the central axis of the shaft. It will be denoted by the symbol J, and is equal to $\dfrac{\pi d^4}{32}$ (see Art. 162, Appendix II). Hence,

$$\text{Resisting moment} = T_r = \frac{s_s J}{c},$$

in which the subscript on s_s is omitted since it will be understood that s_s is the shearing unit-stress on the fiber at the distance c from the center, where c may be any value from O to the radius of the shaft, but since the maximum value of s_s is usually wanted the value of c will usually be the radius of the shaft.

The Torsion Formula.—As already shown in Art. 24, the resisting moment holds the external or twisting moment in equilibrium and is therefore numerically equal to the twisting moment. Thus

$$T = T_r$$

and hence

$$T = \frac{s_s J}{c} \quad \text{or} \quad s_s = \frac{Tc}{J}, \qquad (33)$$

which is called the *torsion formula.*

If s_s is expressed in pounds per square inch T must be expressed in pound-inches, c in inches and J (equal to $\dfrac{\pi d^4}{32}$, Art. 165) in inches to the fourth power (in.^{4}).

Limitations of the Torsion Formula.—The torsion formula is not applicable unless the following conditions are satisfied:

1. The shaft is a circular cylinder, either solid or hollow.
2. The loads lie in a plane or planes perpendicular to the axis of the shaft.

3. The shearing unit-stress does not exceed the shearing proportional limit of the material.

ILLUSTRATIVE PROBLEM

Problem 53. A shaft (Fig. 56) having a diameter, d, of 1.5 in. rotates at constant speed. It has keyed to it a driving pulley A and two driven pulleys B and C. The effective pull for each belt (difference between tensions on the two sides of a pulley) is 40 lb. per in. of width of belt. The belt on pulley A is 6 in. wide, and on B 4 in. wide. Pulley A has a diameter of 36 in., B of 12 in., and C of 18 in. Find the maximum shearing unit-stress in the shaft between A and B; also between B and C.

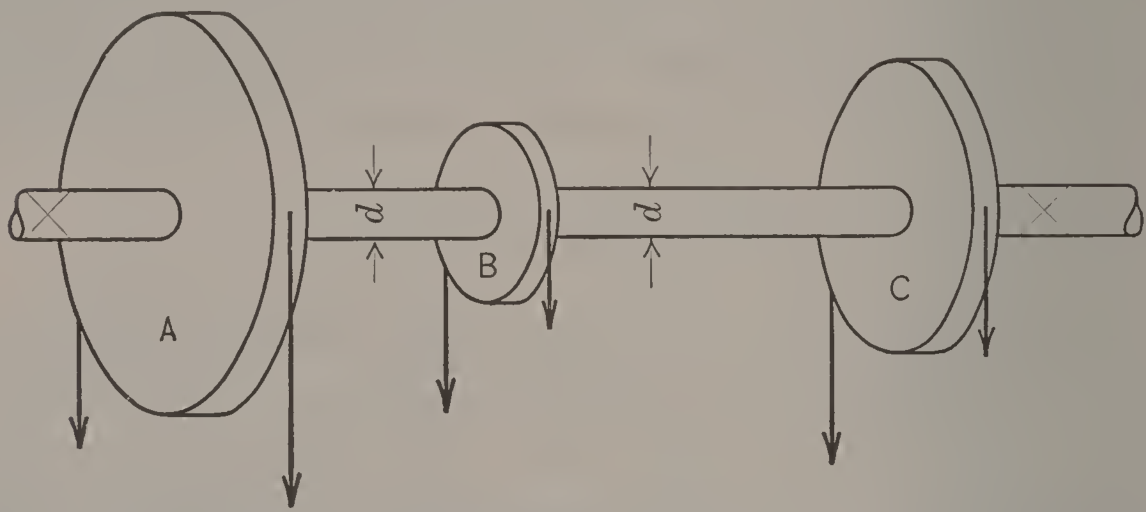


FIG. 56.—Shaft transmits twisting moment from pulley A to pulleys B and C.

Solution.—The twisting moment for any section between A and B is

$$T_1 = 40 \times 6 \times 18 = 4320 \text{ lb.-in.}$$

The maximum shearing unit-stress, s_s', in any section between A and B is

$$s_s' = \frac{T_1 c}{J} = \frac{T_1 \frac{d}{2}}{\frac{\pi d^4}{32}} = T_1 \frac{16}{\pi d^3}$$

$$= 4320 \times \frac{16}{\pi (1.5)^3} = 6520 \text{ lb. per sq. in.}$$

Similarly, the maximum shearing unit-stress, s_s'', at any section between B and C is

$$s_s'' = \frac{T_2 c}{J} = (T_1 - 4 \times 40 \times 6) \times \frac{c}{J}$$

$$= 3360 \times \frac{16}{\pi d^3} = 5070 \text{ lb. per sq. in.}$$

PROBLEMS

54. If a twisting moment of 10,000 lb.-ft. is applied to a cylindrical shaft 4 in. in diameter what shearing unit-stress will be developed in the outer fibers of the shaft? *Ans.* $s_s = 9550$ lb. per sq. in.

55. The shearing yield-point of a certain steel shaft is 30,000 lb. per. sq. in. and a working stress equal to one-third of the yield-point is specified. What should be the diameter of the shaft if the shaft is subjected to a twisting moment of 1200 lb.-in.? *Ans.* $d = 0.85$ in.

56. A workman sometimes breaks a bolt in tightening the nut. If the workman uses a wrench with a handle which gives him a moment arm of 15 in. and he exerts a force of 75 lb. on the handle, what should be the minimum diameter of a steel bolt to prevent the proportional limit of the bolt from being exceeded? Neglect friction.

57. A plate *A* (Fig. 57) is riveted to a fixed member *B* by means of four $\frac{3}{4}$-in. rivets as shown in the figure. Find the shearing unit-stress at the center of each of the rivet areas due to the 2000-lb. loads. Is the torison formula applicable to this problem? Note that the shearing area may be assumed to be two concentric annular areas, equivalent to the cross-sections of two concentric hollow thin-walled cylinders.

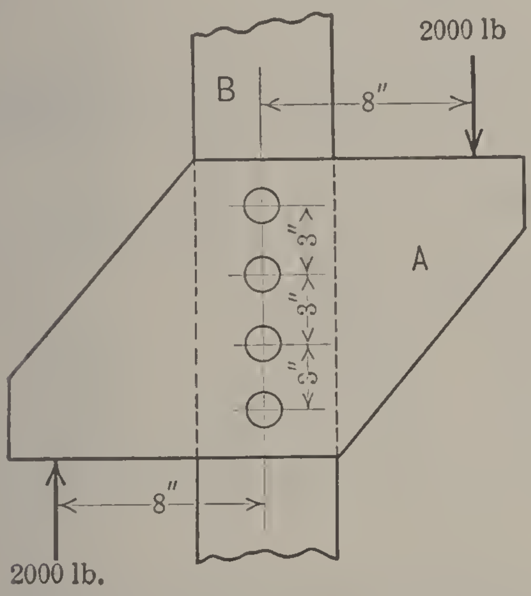


FIG. 57.—Rivets subjected to torsional shear.

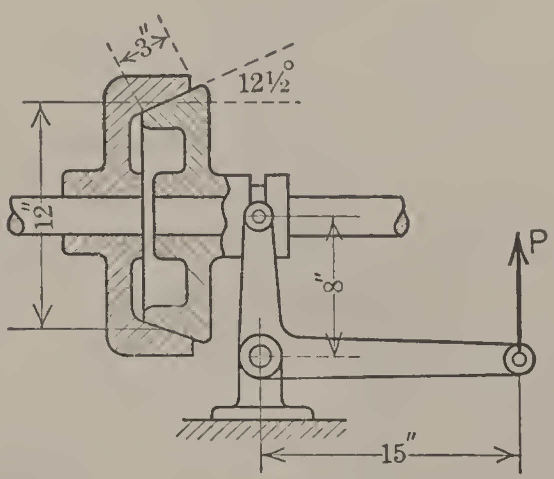


FIG. 58.—Friction clutch transmits twisting moment from one shaft to another.

58. The stern end of a marine propeller-shaft has a diameter of 11 in. and a maximum shearing stress of 8000 lb. per sq. in. To this is connected a hollow steel tail shaft in which the maximum shearing stress is 10,000 lb. per sq. in. The internal diameter is one-half of the external diameter. Find the diameters. *Ans.* $d_1 = 5.21$ in., $d_2 = 10.4$ in.

59. A force *P* of 196 lb. (Fig. 58) will cause a normal pressure of 15 lb. per sq. in. on the rubbing surfaces of the friction clutch shown. If the coefficient of friction is 0.25 and the diameter of the shaft is 2 in., find the maximum shearing unit-stress in the shaft.

60. In the band brake shown in Fig. 59 the force P is 100 lb., the angle of contact, α, is 270°, and the coefficient of friction μ, is 0.2. Therefore, the

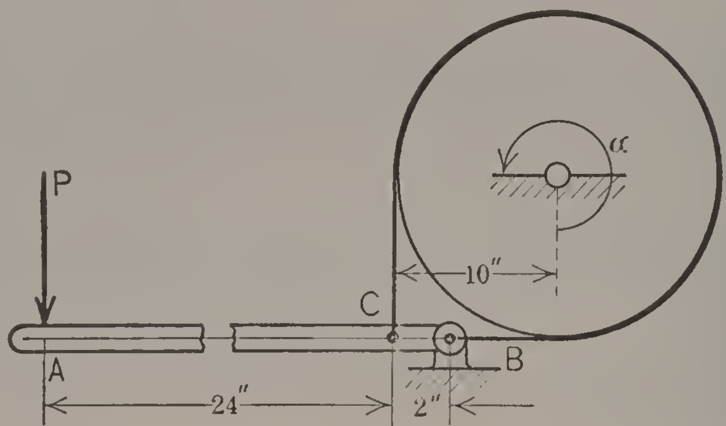


FIG. 59.—Band brake transmits twisting moment to shaft.

tension in the band at C is $T_c = 1300$ lb. and, if the maximum friction is being developed, the tension in the band at B is $T_B = T_A \times e^{\mu\alpha} = 3330$ lb. (*a*) If the diameter of the shaft is 3 in. what is the maximum shearing unit-stress in the shaft? *Ans.* $s_s = 3830$ lb. per sq. in.

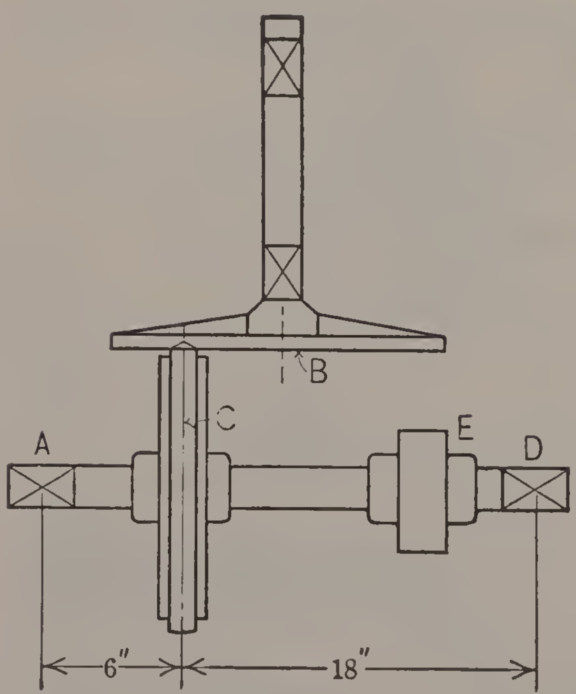


FIG. 60.—Twisting moment exerted on shaft by means of friction drive.

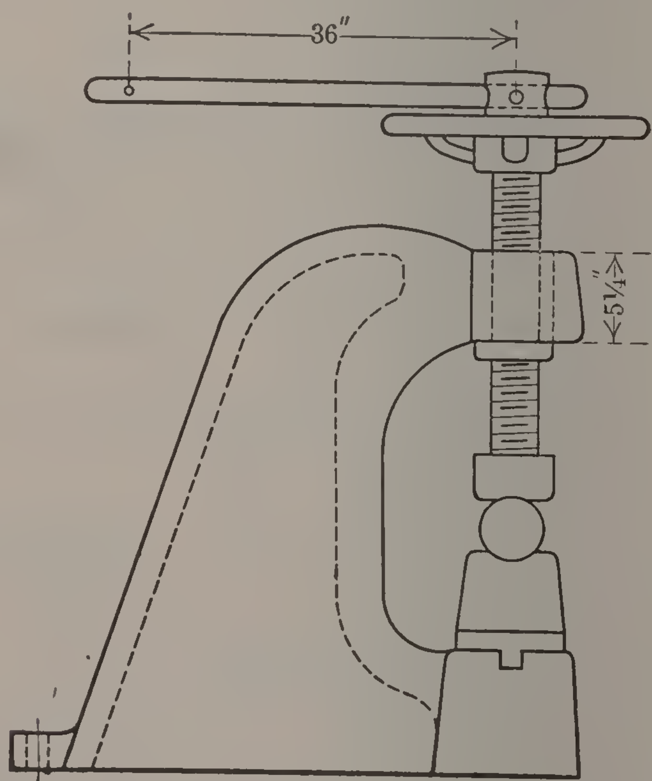


FIG. 61.—Screw of press subjected to torsion due to pull on handle.

61. Compare the strengths of the following shafts when stressed to their limits of proportionality. The first is solid 21 in. in diameter and the shearing proportional limit is 25,000 lb. per sq. in.; the second is hollow, having an outside diameter of 18 in. and an inside diameter of 9 in., and a proportional limit of 45,000 lb. per sq. in.

62. A crown friction drive as shown in Fig. 60 is used on screw-power presses, motor trucks, etc. The cast-iron disk B rotates at 1000 r.p.m. and drives the crown wheel C, which in turn drives the chain sprocket E. The disk C is faced with leather-fiber for which the coefficient of friction is 0.35. If the diameter of the disk C is 20 in. and the normal pressure of B against C is 343 lb. what should be the diameter of the shaft between C and E in order to prevent the shearing unit-stress due to the twisting moment from exceeding 8000 lb. per sq. in. (The working stress is taken low because the load is not a steady or static load.) *Ans.* $d = 0.915$ in.

63. The shaft-straightening hand press shown in Fig. 61 is used for bending or straightening steel shafts requiring a pressure, Q, of about 24,000 lb. The operating screw has a root diameter of 2 in. and there are four square threads per inch. If the coefficient of friction is 0.2, a force of 161 lb. applied on the handle at A will cause the required force Q on the shaft. Calculate the maximum shearing unit-stress in the shaft.

27. Twisting Moment in Terms of Horsepower and Speed.—Frequently the forces acting on the shaft are unknown, and the twisting moment must be found from the horsepower transmitted by the shaft and the speed (angular velocity) of the shaft. This may be done as follows: Let a pulley A (Fig. 62) be keyed to a shaft B and transmit a twisting moment T (equal to Pp) to the shaft when rotating at a constant angular velocity of ω radians per second or n revolutions per minute. The work done by the twisting couple T, in any angular displacement is the product of the moment of the couple and the angular displacement (in radians). Hence,

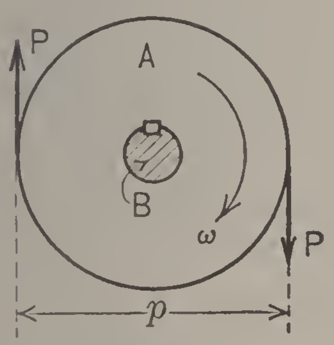


FIG. 62.

$$\text{Work done on shaft per second} = T\omega$$
$$= T\frac{2\pi n}{60}, \text{ since } \omega = \frac{2\pi n}{60} \text{ radians.}$$

If the twisting moment T is expressed in inch-pounds the work done will also be expressed in inch-pounds. Now the rate of doing work at 550 ft.-lb. per sec. or 550×12 in.-lb. per sec. is defined to be a horsepower, and hence the number of horsepowers (h.p.) transmitted by the shaft is

$$(\text{h.p.}) = \frac{T 2\pi n}{60 \times 550 \times 12}$$

in which T is expressed in lb.-in. and n in r.p.m. Hence

$$T=\frac{60\times550\times12\times(\text{h.p.})}{2\pi n}\text{ in.-lb.}, \quad . \quad . \quad . \quad . \quad . \quad (34)$$

which expresses the relation between twisting moment, horse-power, and speed.

PROBLEMS

64. The hydraulic turbines in the Keokuk water-power plant are rated at 10,000 h.p., with an overload capacity of 13,500 h.p. The vertical shaft connecting the turbine and the generator is 25 in. in diameter and rotates at 57.7 r.p.m. What is the maximum shearing unit-stress developed in the shaft when developing full load and when developing maximum overload?

65. A pulley 10 ft. in diameter is mounted on a $2\frac{1}{2}$-in. shaft and rotates at 150 r.p.m. transmitting power to a belt. If the belt tensions are 1800 lb. and 1400 lb. (*a*) what horse-power is transmitted ? (*b*) What is the maximum fiber stress in the shaft ? *Ans.* (*a*) (*h.p.*) = 114, (*b*) $s_s = 5220$ lb. per sq. in.

66. A hollow shaft is used to transmit 6000 h.p. The outside diameter is 18 in., the speed is 90 r.p.m., and the maximum shearing unit-stress developed in the shaft is 12,000 lb. per sq. in. Find the inside diameter of the shaft.
Ans. $d_1 = 16.5$ in.

67. What horsepower can a solid steel shaft 6 in. in diameter transmit when rotating at 120 r.p.m. and developing a shearing unit-stress of 10,000 lb. per sq. in. in the outer fiber?

28. Angle of Twist of Cylindrical Shaft.—In some problems the size of shaft required to transmit a given twisting moment (or given horsepower at a given speed) is controlled by the allowable angle of twist rather than the allowable shearing unit-stress. In other words, stiffness rather than strength is the controlling factor in the design. The angle of twist, θ, (Fig. 54) caused by a given twisting moment T may be found as follows:

As explained in Art. 5,

$$\frac{s_s}{\epsilon_s}=\text{a constant}=E_s,$$

provided that s_s does not exceed the shearing proportional limit of the material. But as shown in Arts. 25 and 26

$$s_s=\frac{Tc}{J} \quad \text{and} \quad \epsilon_s=\frac{c\theta}{l}.$$

Therefore,

$$E_s = \frac{\frac{Tc}{J}}{\frac{c\theta}{l}} = \frac{Tl}{J\theta}$$

Hence, the angle of twist, θ, when not greater than that corresponding to the proportional limit of the material is,

$$\theta = \frac{Tl}{E_s J}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

in which θ is expressed in radians, any consistent units of force and length being used to express the other quantities. Thus, T is usually expressed in inch-pounds, l in inches, E_s in pounds per square inch, and J in inches.[4]

PROBLEMS

68. A steel shaft 4 in. in diameter transmits 200 h.p. at a speed of 250 r.p.m. The distance between the driving and driven pulleys is 10 ft. Determine whether the following two requirements are satisfied:

(*a*) Maximum shearing unit-stress not to exceed 10,000 lb. per sq. in.

(*b*) Twist of shaft not to exceed 1 degree per 20 diameters of length.

Ans. $s_s = 4020$ lb. per sq. in. $\theta = 0.77°$ per 20 diam.

69. A wrought-iron shaft 7.5 ft. long and 2 in. in diameter twists through an angle of 10.5 degrees when resisting a twisting moment of 2500 lb.-ft. The shearing proportional limit of the material is 24,000 lb. per sq. in. Find the value of the shearing modulus of elasticity.

Ans. $E_s = 9{,}360{,}000$ lb. per sq. in.

70. A hollow bronze shaft having an outside diameter of 6 in. and an inside diameter of 4 in. is 12 ft. long. If the shaft is twisted by applying moments at its ends, what will be the angle of twist when the maximum shearing unit-stress is 8000 lb. per sq. in. ($E_s = 6{,}000{,}000$ lb. per sq. in.)

71. The steel shaft connecting a hydraulic turbine with an electric generator was so long that it twisted enough to cause electrical trouble in the generator. If the shaft had been replaced by one with the same diameter but made of steel having a greater shearing strength would the amount of twist have been decreased?

29. Shaft Couplings.—Two shafts frequently are connected by means of a bolted coupling (Fig. 63*a*) so that, as the twisting moment is transmitted from one shaft to the other, shearing stresses are developed in the bolts. For example, let the shaft A (Fig. 63*a*) exert a twisting moment on the shaft B just sufficient to overcome the resistance to motion of shaft B, the resisting moment exerted by B being equal to the moment of the shearing stresses in the bolts as indicated in Fig. 63b.

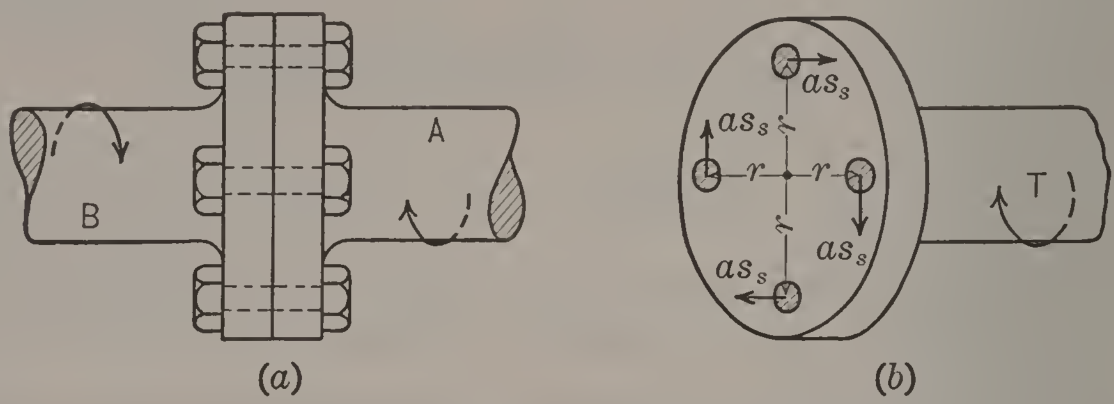


FIG. 63.—Shear in bolts of coupling.

Now if the diameter, d, of the bolts is small in comparison with the distance, r, from the center of the bolts to the center of the shaft, the shearing unit-stress, s_s, over the cross-section area, a, of each bolt may be assumed to be constant. Hence the resisting moment, T_r, is

$$T_r = m(as_s \cdot r)$$

$$= m\frac{\pi d^2}{4}s_s \cdot r,$$

in which m is the number of bolts, all of which are on a circle of radius r. And since this resisting moment is equal and opposite to the twisting moment T, then

$$T = m\frac{\pi d^2}{4}s_s \cdot r. \qquad (36)$$

If T is expressed in inch-pounds, d and r must be expressed in inches, and s_s in pounds per square inch. But from Art. 27

$$T = \frac{60 \times 550 \times 12 \times (\text{h.p.})}{2\pi n} \text{ in.-lb.}$$

Therefore,

$$\frac{60 \times 550 \times 12 \times (\text{h.p.})}{2\pi n} = m\frac{\pi d^2}{4}s_s \cdot r \qquad (37)$$

from which any one of the quantities may be found if all the others are known.

The resisting moment, T_r, could have been obtained from the expression $\frac{s_s J}{c}$ in which

$$J = m(\bar{J} + ar^2), \text{ see Art. 164}$$

but $\bar{J} = \frac{1}{32}\pi d,^4$ and since d is small in comparison with r then d^2 is negligible in comparison with r^2. Hence the resisting moment is

$$T_r = \frac{s_s mar^2}{r} = m\frac{\pi d^2}{4} r s_s$$

which is the same expression as found by the method used above.

PROBLEMS

72. A solid shaft 4 in. in diameter is to be connected to another shaft of the same size by means of a coupling as shown in Fig. 63*a*. If six $\frac{3}{4}$-in. bolts are used on a circle having a diameter of 10 in., what (h.p.) can the shaft transmit when rotating at 150 r.p.m. and when the shearing unit-stress in the bolts is 10,000 lb. per sq. in. What will be the maximum shearing unit-stress in the shaft?

73. A coupling is to be used to connect two shafts having diameters of 4 in. The maximum allowable shearing unit-stress in the shafts is 10,000 lb. per sq. in., the diameter of the bolt circle is 8 in. and the allowable shearing unit-stress in the bolts is 9000 lb. per sq. in. Find the number of $\frac{3}{4}$-in. bolts necessary. *Ans.* Eight.

30. Stress Beyond Proportional Limit. Modulus of Rupture.—As shown in Art. 26 the value of s_s in the torsion formula, $T = \frac{s_s J}{c}$, is the shearing unit-stress at the distance c from the axis of the shaft *only when the proportional limit of the material is not exceeded.* If the twisting moment, T, causes a unit-stress greater than the shearing proportional limit of the material the internal or resisting moment is still equal to (holes in equilibrum) the external or twisting moment, but the resisting moment is not given by the expression $\frac{s_s J}{c}$ since this expression is found by assuming that the shearing unit-stress varies directly as the distance from the center of the shaft which is true only when the stress does not exceed the proportional limit of the material.

This fact is illustrated in Fig. 64. A diameter before twisting is assumed to be a diameter (straight line) after twisting even though the material is stressed beyond the proportional limit, and hence the unit-strain in each of the two shafts (Fig. 64), varies directly as the distance from the center of the shaft. But in the shaft that is stressed beyond the proportional limit (Fig. 64*b*) the unit-stress is not proportional to the unit-strain except near the center of the shaft where the strain is small, and hence the unit-stress does not vary directly as the distance from the center of the shaft except for a short distance out from the center. If the material of the shaft is ductile enough to have a yield-point the unit-stress will vary about as shown in Fig. 64(*b*) when the yield-point of the shaft is reached.

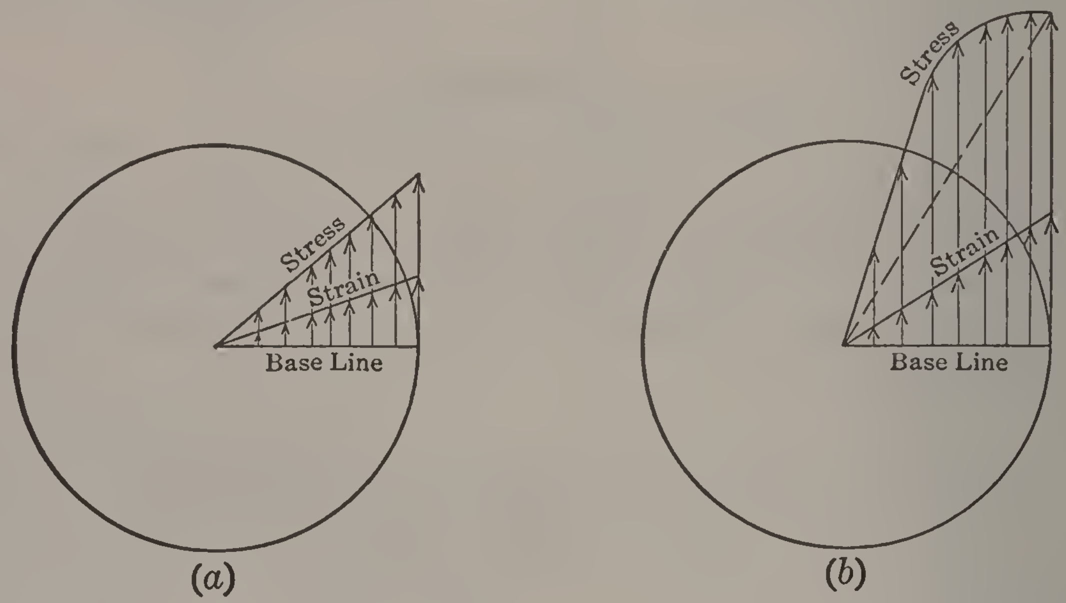


FIG. 64.—Distribution of shearing stress in shaft when stressed above proportional limit.

Modulus of Rupture.—Now the moment of the stresses in Fig. 64(*b*) is larger, for a given unit-stress at the surface of the shaft, than it would be if the unit-stress varied directly from zero at the center to this given value at the surface as indicated by the dotted line. Therefore, if the shearing ultimate strength[3] of the material is substituted for s_s in the torsion formula $\left(T=\frac{s_sJ}{c}\right)$ the value found for T would be smaller than the maximum twisting moment that can be applied to the shaft. Tests show that the maximum twisting moment that can be applied to cylindrical steel bars is from 10 to 20 per cent greater than that calculated from the torsion formula by substituting the shearing ultimate strength for s_s.

The value of s_s found from the torsion formula by substituting

[3] The shearing yield-point of a ductile material is regarded as its useable shearing ultimate strength.

for T the value of the maximum twisting moment that a shaft resists when tested to rupture is called the shearing *modulus of rupture* and will be denoted by s_r. Thus

$$s_r = \frac{T_{\max.}\, c}{J}.$$

It should be noted that s_r is not the unit-stress in the material caused by the moment $T_{\max.}$, and, as already noted, it is not equal to the shearing ultimate strength [4] of the material; it is merely a value (expressed in lb. per sq. in.) from which the maximum twisting moment that a cylindrical shaft can resist, may be found.

31. Torsion of Non-circular Sections.—As stated in Art. 26 the torsion formula, $T = \frac{s_s J}{c}$, applies only to a shaft having a circular cross-section, in which case a plane section of the shaft before twisting is a plane section after twisting and the shearing unit-stress varies directly as the distance from the center, the unit-stress being maximum at the outer fiber.

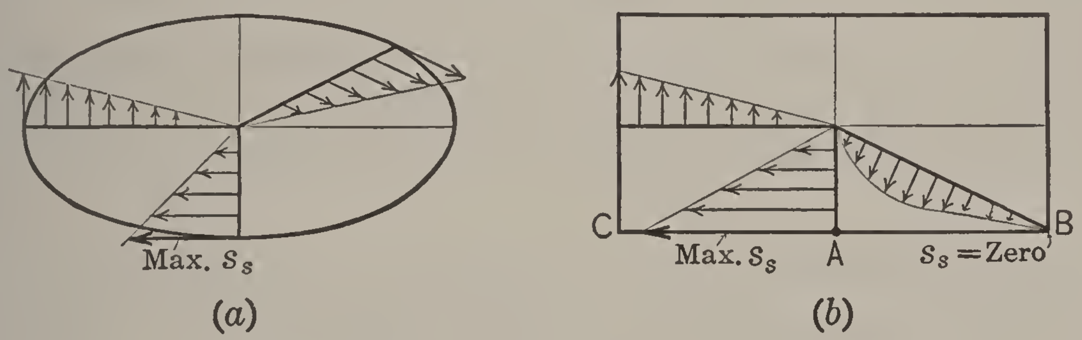


FIG. 65.—Stress in shafts having non-circular sections.

If a bar has an elliptical, rectangular, or other non-circular cross-section, a plane section before twisting becomes a warped surface after twisting, and, if the section is free from re-entrant angles, the maximum unit-stress occurs on the surface fiber *nearest* to the center of the shaft instead of the fiber most remote from the center. The distributions of stress on the cross-sectional areas of elliptical and rectangular bars are shown in Fig. 65.

The derivation of the equations giving the maximum shearing unit-stress and angle of twist for bars that have non-circular cross-sections is beyond this scope of this book.

[4] For a method of finding the shearing ultimate strength of a material from a torsion test of a solid cylindrical specimen see Upton's Materials of Construction (John Wiley & Sons), p. 52.

Torsion of Members Other than Shafts.—Frequently structural members such as I-beams, flat plates, columns, etc., in buildings, ships, locomotives, cranes, etc., are subjected to twisting actions, and although the twisting in such members is, in general, considered to produce secondary effects, the twisting may have a marked influence on the stiffness and strength of the member, particularly since members that are designed to resist bending have, in general, relatively small resistance to torsion; torsional action in structural members should therefore be reduced to a minimum, or allowance should be made for the effects of the torsion.

CHAPTER V

TRANSVERSE LOADS. STRESSES IN STATICALLY DETERMINATE BEAMS

32. Preliminary Considerations.—A *beam* is a bar that is bent by forces acting perpendicular to the axis of the bar.[1] Forces that act on a beam are called *transverse* or *cross-bending* loads. It will be assumed that the cross-bending loads acting on a beam lie in one plane containing the central longitudinal axis of the beam and that this plane is a plane of symmetry of the beam.

Beams are important members in many engineering structures and machines. Several common types of beams are described as follows:

A *simple* beam is one that rests on two supports at the ends of the beam. Fig. 66 shows a horizontal simple beam subjected to two equal concentrated loads at the third-points.

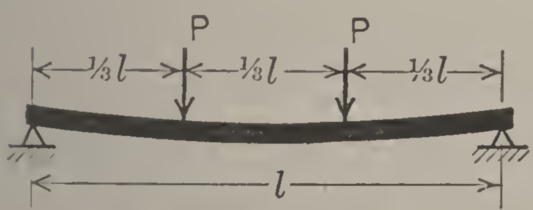


Fig. 66.—Simple beam.

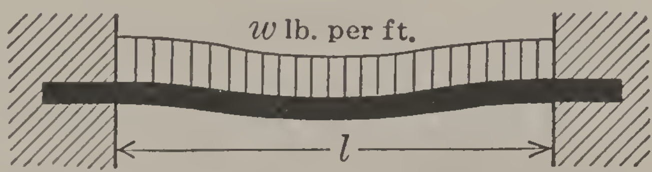


Fig. 67.—Fixed beam.

A *fixed* or *fixed-ended* beam is one so restrained at its ends that the slope of the curve of the beam at the restrained ends does not change when the load is applied. Fig. 67 shows a beam fixed at both ends, and subjected to a uniformly distributed load of w pounds per foot; Fig. 68 shows a beam fixed at one end and supported at the other, and subjected to a concentrated load at mid-span. The end connections of beams in structures and machines

[1] If the beam acts as it is assumed to act, in this and the following chapters, its length must be at least several times its depth. Further, the thickness or width of the beam must be sufficient to prevent collapse by wrinkling; thus, extremely deep and extremely thin beams are excluded.

frequently offer considerable restraint, but not enough that the beams may be considered "fixed"; such beams are then intermediate between simple beams and fixed beams.

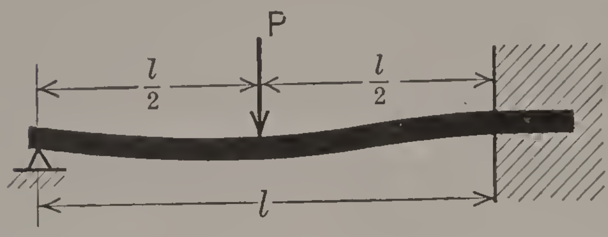


FIG. 68.—Beam fixed at one end, supported at other end.

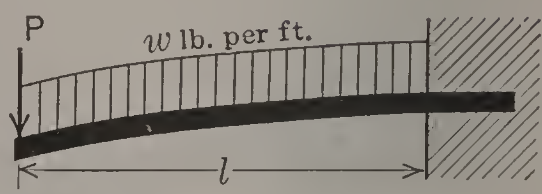


FIG. 69.—Cantilever beam.

A *cantilever* beam is one that is fixed at one end and free at the other. Fig. 69 shows a cantilever beam subjected to a uniformly distributed load over its entire length, and to a concentrated load at its end.

Fig. 70 shows an *overhanging* beam that overhangs both supports and that carries concentrated loads at the ends of the beam and a uniformly distributed load over the span between supports.

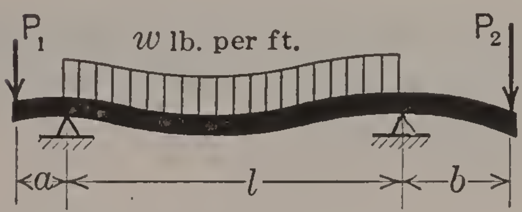


FIG. 70.—Overhanging beam.

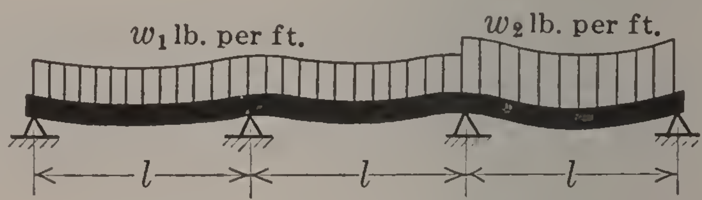


FIG. 71.—Continuous beam.

A *continuous* beam is one that rests on more than two supports. Fig. 71 shows a continuous beam with three equal spans carrying a uniformly distributed load of w_1 pounds per foot over two spans and a uniformly distributed load of w_2 pounds per foot over the third span.

Pure bending is bending caused by couples. Thus the mid-third of the beam in Fig. 66 is subjected to pure bending. Bending produced by forces that do not form couples is called *ordinary* bending. As will be shown later, a beam subjected to pure bending has no shearing stresses developed in it; the stresses on any section are normal (tensile or compressive) stresses whereas in *ordinary* bending the loads develop shearing stresses as well as normal stresses.

Reactions.—For convenience the forces exerted on a beam by the supports are called *reactions* and the other forces are called loads, but both loads and reactions are merely external forces that act on the beam and hold the beam in equilibrium.

If the loads are known, the reactions can be found by use of the equations of equilibrium *provided that there are not more than two reactions.* For, the forces that hold the beam in equilibrium constitute a parallel force system in a plane, and for such a force system there are only two independent equations of equilibrium; namely,

$$\begin{array}{ccc} \Sigma F=0 & & \Sigma M_A=0 \\ & \text{or} & \\ \Sigma M=0 & & \Sigma M_B=0 \end{array}$$

which state that the algebraic sum of the forces and the algebraic sum of the moments of the forces about any point are equal to zero. Or, the algebraic sum of the moments of the forces about each of two points A or B, not in a line parallel to the forces, are equal to zero.

Statically Determinate and Statically Indeterminate Beams.—Beams for which the reactions can be found from the equations of equilibrium are called *statically determinate* beams, and those for which the number of unknown reactions is greater than the number of equilibrium equations are called *statically indeterminate beams.* Simple and cantilever beams, and overhanging beams that rest on two supports, are *statically determinate* beams, whereas, fixed-ended beams and continuous beams are *statically indeterminate,* and hence require equations in addition to the equilibrium equations in order to determine the reactions.

This chapter deals mainly with the problem of determining the relation between the external forces acting on statically determinate beams and the stresses which the forces develop in the beams. Statically indeterminate beams are discussed in Chapters VII and IX.

PROBLEMS

74. Find the reactions of the supports for the beam shown in Fig. 72; neglect the weight of the beam.

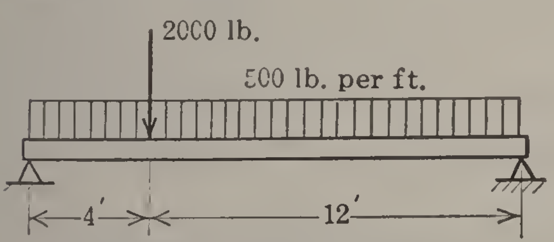


FIG. 72.

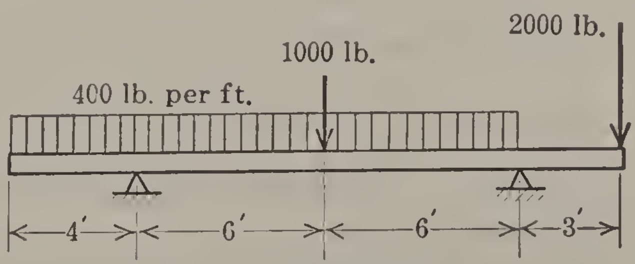


FIG. 73.

75. Find the reactions of the supports for the beam shown in Fig. 73; neglect the weight of the beam. *Ans.* $R_1 = 4267$ lb. $R_2 = 5133$ lb.

33. Vertical Shear, Resisting Shear, Bending Moment, and Resisting Moment.—In Fig. 74(a) is shown a simple beam subjected to two concentrated loads, the weight of the beam being negligible. The reactions R_1 and R_2 are found to be 6500 lb. and 3500 lb., respectively. Let it be required to determine the character of the internal forces (stresses) that must occur on any section of the beam, such as section AA. If the portion of the beam to the right of section AA is removed, forces equivalent to those that the right portion exerted on the left portion must be applied on section AA in order to hold the left portion in equilibrium, since it was in equilibrium before the right portion was

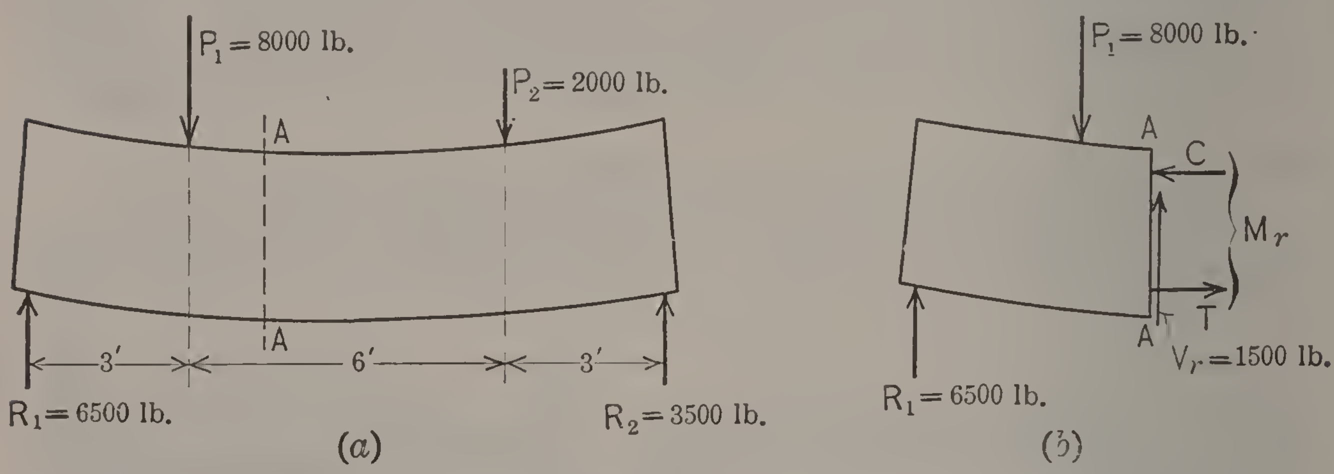


FIG. 74.—Stresses at any section hold external forces in equilibrium.

removed. The problem, therefore, is to determine the forces (stresses) that must act on section AA in order to hold the forces R_1 and P_1 in equilibrium.

Now, it is only the unbalanced part, or resultant, of the external forces that must be held in equilibrium by the stresses on the section of the beam, and hence, it is convenient to have a name for the resultant, or stress-producing part, of the loads. Thus,

> The *vertical shear* for a section of a beam is the magnitude of the resultant of the forces (loads and reactions) that lie on one side of the section. Or, in other words, it is the algebraic sum of the forces that lie on one side of the section.

The symbol V will be used to denote vertical shear and, for convenience, the forces that lie to the *left* of the section will, as a rule, be used.

The total shearing stress on a section of the beam is called the *resisting shear* at the section and will be denoted by the symbol V_r. And since, from the condition of equilibrium, the algebraic sum of all the forces acting on the left (or right) portion of the beam must equal zero we have,

$$\text{Vertical shear} = \text{Resisting shear}$$

or,

$$V = V_r.$$

For the beam shown in Fig. 74*a* the vertical shear V for section AA is a downward force equal to 8000 lb.—6500 lb. or 1500 lb., and hence the resisting shear, V_r, on section AA (Fig. 74*b*) is an upward stress of 1500 lb.

Bending Moment and Resisting Moment.—The left portion of the beam (Fig. 74*b*), however, will not be in equilibrium unless the algebraic sum of the moments of the forces acting on it are also equal to zero. Now the moment of R_1 about a line in the section AA is greater than the moment of P_1, and hence this unbalanced moment of the external forces must be resisted by a moment exerted by the forces acting on the section AA. These forces on the section AA are the normal stresses exerted by the right portion of the beam on the left portion and consist of compressive stresses on the upper part of the section and tensile stresses on the lower part.

Since only the unbalanced part of the moments of the external forces is effective in producing tensile and compressive stresses in the beam, it is convenient to have a name for this stress-producing moment. Thus,

> The *bending moment* at a section of a beam is the moment, about the section, of the resultant of the external forces that lie on one side of the section; that is the algebraic sum of the moments, about the section, of the external forces that lie on one side of the section.

The bending moment will be denoted by the symbol M and the forces that lie to the *left* of the section will, as a rule, be used. The bending moment at section AA (five feet from R_1) is,

$$M = 6500 \times 5 - 8000 \times 2 = 16{,}500 \text{ lb.-ft.}$$

Now, as noted above, the unbalanced moment of the external forces (bending moment) about the section AA is resisted by the moment of the tensile and compressive stresses on the section. This moment is called the *resisting moment.* Hence,

> The resisting moment at a section of a beam is defined to be the algebraic sum of the moments of the stresses acting on the section, about a line in the section. It will be denoted by the symbol M_r.

But from the condition of equilibrium the sum of the moments of all the forces acting on the left portion of the beam (Fig. 74*b*) must equal zero. Therefore,

Moment of external forces = Moment of internal forces;

that is,

Bending moment = Resisting moment

or

$$M = M_r.$$

Further, from the condition of equilibrium, the sum of all the horizontal forces acting on the left portion of the beam (Fig. 74*b*) must equal zero. And since no horizontal external forces act on the beam, we have

Algebraic sum of the horizontal stresses = 0.

That is, the sum of the compressive stresses, denoted by C (Fig. 74*b*), equals the sum of the tensile stresses, denoted by T. Hence, $T = C$.

Summarizing: Since any portion of the beam (Fig. 74*b*) is in equilibrium the forces acting on the portion must satisfy the following equations:

Algebraic sum of horizontal stresses = 0, or $T = C$
Algebraic sum of vertical forces = 0, or $V = V_r$
Algebraic sum of moments of forces = 0, or $M = M_r$. (38)

An expression will now be found for the resisting moment M_r in terms of the tensile or compressive unit-stress at any point in the cross-section of the beam, and the dimensions of the cross-section. The shearing unit-stress in the beam will be discussed in Art. 40.

In discussing the stresses in a beam it is convenient to consider the beam to be composed of fibers, a *fiber* being a rod having an elementary cross-sectional area da and extending the length of the beam parallel to the axis of the beam; a *fiber unit-stress* then is the intensity of stress on a section of any fiber, and in general may be either a normal or a shearing unit-stress although for the present the normal unit-stress only will be considered.

34. Expression for Resisting Moment. The Flexure Formula.—In order to find the sum of the moments of the stresses on the fibers at any section of the beam, that is, in order to find the resisting moment at any section, the way in which the fiber unit-stress varies with the position of the fiber in the beam must be known, that is, the law of distribution of the intensity of stress over the section must be known.

Distribution of Stress on Section.—In order to show how the unit-stress on a fiber varies with the position of the fiber in the beam there is required a knowledge of:

(a) The way in which the strain of a fiber varies with the position of the fiber in the beam when the beam is bent, and

(b) The relation, for the material of which the beam is made, of the unit-stress in a fiber to the strain of the fiber.

Information concerning these points comes mainly from the results of experiments:

(a) When a simple horizontal beam is bent the fibers on the top side shorten and those on the bottom elongate, and the fibers in one plane within the beam do not deform. This plane is called the *neutral surface*, and the line of intersection of the neutral surface and a cross-section of the beam is called the *neutral axis* for the section. Further, when a beam is subjected to bending, experiments show that if two straight lines, DE and FG (Fig. 75) are drawn, before the beam is bent, on the side of the beam a short distance, MN, apart, these lines will still be approximately

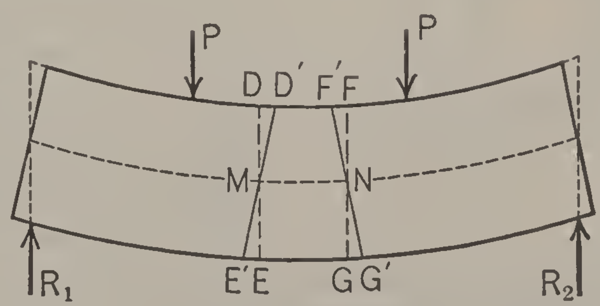


FIG. 75.—Strain of fibers in a beam.

straight lines, $D'E'$ and $F'G'$, after the beam is bent.[2] Thus, one fiber MN, in the neutral surface, remains constant in length and

> The strain of any fiber is directly proportional to the distance of the fiber from the neutral surface. Further, since the original lengths of all the fibers are equal[3] the unit-strain of any fiber is also proportional to the distance of the fiber from the neutral surface.

(*b*) Now if one of the fibers were removed from the beam and subjected to an axial load in a testing machine so that the unit-stress $\left(\text{equal to } \frac{P}{a}\right)$ and the unit-strain $\left(\text{equal to } \frac{e}{l}\right)$ could be measured it would be found, as stated in Art. 5, that the *unit-stress* on the fiber is proportional to the unit-strain of the fiber *provided that the proportional limit of the material is not exceeded.* Further, it is assumed that the fiber when in the beam acts according to the same law as when tested alone.

Therefore, since the unit-strain of a fiber is directly proportional to the distance of the fiber from the neutral surface, and since the unti-stress on a fiber is directly proportional to the unit-strain of the fiber, it follows that

> The unit-stress on a fiber of a beam at any section of the beam is directly proportional to the distance of the fiber from the neutral axis.

[2] Since the straight lines may be considered to be traces of cross-sectional planes, this fact is sometimes stated: "A plane section before bending is a plane section after bending;" the essential fact, however, is not that a plane section is conserved, but that a fiber changes its length and that this longitudinal strain is proportional to the distance of the fiber from the neutral surface. The selection of *plane* sections is mainly a matter of convenience in explanation. If shearing stresses exist in the beam, two parallel plane sections DE and FG (Fig. 76) before bending become curved surfaces $D'E'$ and $F'G'$ after bending but the longitudinal strains of the fibers are affected very little by the shearing deformation in beams of the usual proportions.

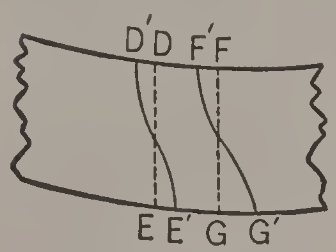


FIG. 76.

[3] If the beam is a curved beam (not straight before it is loaded) the lengths of the fibers between two normal sections of the beam would not be equal and hence, although the total deformation would be proportional to the distance from the neutral axis, the *unit-strain* would not be proportional to this distance. This demonstration is, then, limited to straight beams.

Thus, if s_y and s_c (Fig. 77) denote the unit-stresses on fibers at the distances y and c, respectively, from the neutral axis the above statement is expressed mathematically as follows:

$$\frac{s_y}{s_c}=\frac{y}{c} \quad \text{or} \quad \frac{s_y}{y}=\frac{s_c}{c}=\text{a constant.}$$

Expression for Resisting Moment.—An expression may now be found for the resisting moment (the algebraic sum of the moments of the stresses on the cross-section about the neutral axis) in terms of the unit-stress on any fiber and the dimensions of the cross-section of the beam as follows:

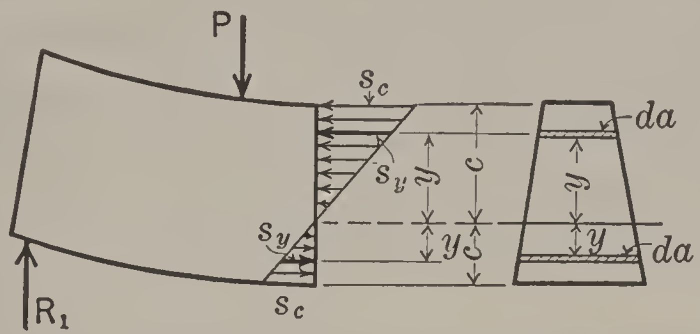


FIG. 77.—Intensity of stress varies directly as distance from neutral axis.

The unit-stress, s_y, on a fiber at the distance y from the neutral axis (Fig. 77) may be assumed to be constant over the cross-sectional area, da, of the fiber, and hence,

Total stress on area of one fiber $= s_y da$

Moment of total stress on one fiber $= s_y y da$

Sum of moments of stresses on all fibers $= M_r = \int s_y y da.$

This may be written

$$M_r = \int \frac{s_y}{y} y^2 da,$$

and since, as shown above, $\frac{s_y}{y}=\frac{s_c}{c}=$ *a constant*, this may be written

$$M_r = \frac{s_y}{y}\int y da \quad \text{or} \quad M_r = \frac{s_c}{c}\int y^2 da.$$

But the expression $\int y^2 da$ is the moment of inertia of the cross-section of the beam with respect to the neutral axis, since y is measured from the neutral axis (see Appendix II.). Therefore,

$$\text{Resisting Moment} = M_r = \frac{sI}{c},$$

in which s is the tensile or compressive unit-stress on a fiber at the distance c from the neutral axis, and since the maximum value of s is usually desired, the distance c will usually be taken as the distance to the most remote fiber.

Flexure Formula.—But as shown in Art. 33, $M = M_r$. Therefore,

$$M = \frac{sI}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

which is called the *flexure formula*. It expresses the relation between the external forces acting on the beam, the tensile or compressive unit-stress at any point in the beam, and the dimensions of the cross-section of the beam. If s is expressed in pounds per square inch as is customary in the United States then M must be expressed in pound-inches, I in inches^4, and c in inches. Further, $\frac{I}{c}$ is called the *section modulus* of the beam and is expressed in inches^3.

Position of the Neutral Axis.—The value of I in the flexure formula cannot be found unless the position of the neutral axis in the area is known. Now, as noted in Art. 33, the sum of the compressive stresses on the section must be equal to the sum of the tensile stress on the section, that is, the sum of all the horizontal stresses at any section must be equal to zero; this condition serves to locate the neutral axis. Thus,

Σ horizontal stresses on section $= 0$,

that is (see Fig. 77),

$$\int s_y da = 0,$$

or

$$\int \frac{s_y}{y} y da = 0,$$

and hence

$$\frac{s_y}{y} \int y da = 0, \qquad \text{since } \frac{s_y}{y} = \text{a const.}$$

But $\frac{s_y}{y}$ is not equal to zero. Therefore,

$$\int y da = 0,$$

in which y is measured from the neutral axis, but $\int y da$ is, by definition, the moment of the cross-sectional area of the beam with respect to the neutral axis (see Fig. 78), and may be written $a\bar{y}$ in which a is the total area of the cross-section and $\bar{y}$ is the distance of the centroidal axis from the neutral axis, about which moments are taken. Thus,

$$\int y da = a\bar{y} = 0.$$

But a is not equal to zero. Therefore,

$$\bar{y} = 0,$$

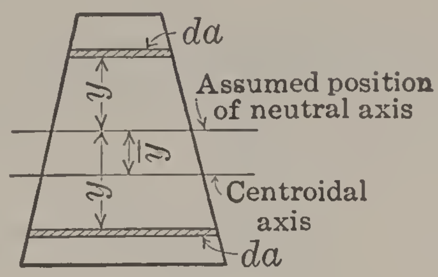


FIG. 78.

which states that the distance from the centroidal axis to the neutral axis is zero, and hence the neutral axis is coincident with the centroidal axis. It is assumed that the student is familiar with the methods of locating the centroids of area; this topic, however, is treated in Appendix I. Further, the methods for determining the moment of inertia of an area with respect to an axis is discussed in Appendix II.

ILLUSTRATIVE PROBLEMS

Problem 76. A simple beam having a rectangular cross-section (Fig. 79) is subjected to a uniformly distributed load of 400 lb. per ft. (including the weight of the beam) over the whole span, and a concentrated load of 2000 lb. at a distance of 4 ft. from the left support. Find the tensile unit-stress on the outer fiber of the beam at the section AB, 5 ft. from the left support.

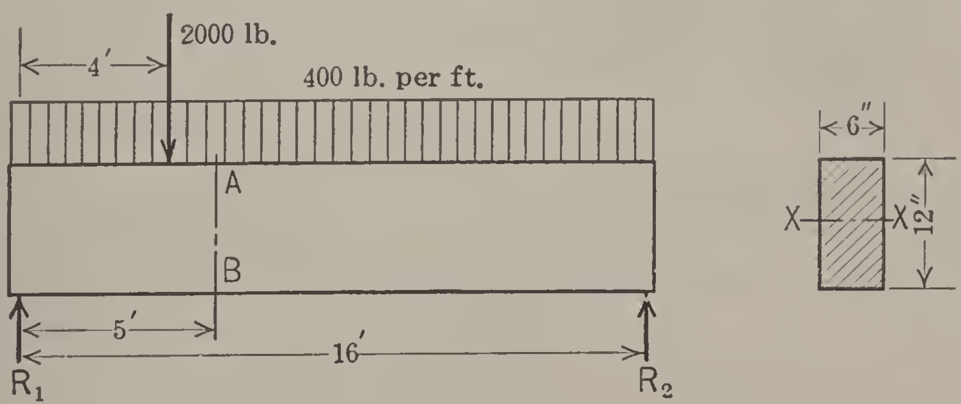


FIG. 79.

Solution.—First Method.—The reactions are found to be

$$R_1 = 4700 \text{ lb.}, \quad R_2 = 3700 \text{ lb.}$$

The bending moment M at the section AB is

$$M = 4700 \times 5 - 400 \times 5 \times 2.5 - 2000 \times 1$$
$$= 16{,}500 \text{ lb.-ft.} = 198{,}000 \text{ lb.-in.}$$

The centroidal axis (and hence the neutral axis) of the cross-section is the central horizontal axis, xx, and the moment of inertia of the cross-section about the neutral axis is (see Art. 165)

$$\bar{I} = \tfrac{1}{12}\,bd^3 = \tfrac{1}{12}\,6(12)^3 = 864 \text{ in}^4.$$

The tensile unit-stress on the bottom fiber at the section AB, then, is

$$s = \frac{Mc}{I} = \frac{198{,}000 \times 6}{864}$$

$$= 1370 \text{ lb. per sq. in.}$$

provided that this unit-stress is not greater than the proportional limit of the material. A compressive unit-stress of the same magnitude occurs on the top fiber.

Second Method of Solution.—Instead of expressing the resisting moment as the algebraic sum of the moments of the stresses on the fibers of the cross-section, which leads to the expression sI/c, the resisting moment may be expressed as the moment of the couple formed by the resultants of the tensile and compressive stresses. Thus as shown in Fig. 80 the action line of the resultant, C, of the compressive stress is $\frac{2}{3}$ of $\overline{OA}$ from O, and similarly the resultant T acts at a distance of $\frac{2}{3}\,\overline{OB}$ from O. Hence the resisting moment is $C \times \frac{2}{3}d$ or $T \times \frac{2}{3}d$. Further, the magnitude of C (and T) is the product of the average unit-stress, $\frac{1}{2}s$, on the area above (or below) the neutral axis and that area. Thus

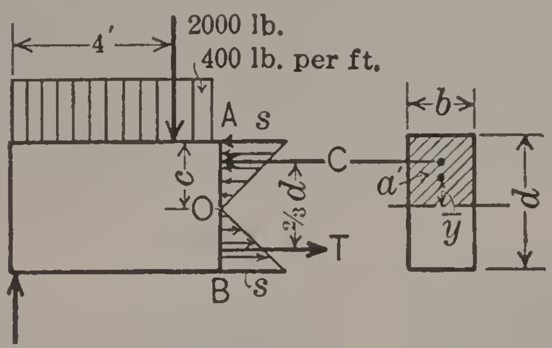


FIG. 80.—Resisting couple.

$$C = T = \tfrac{1}{2}s \times \tfrac{1}{2}bd,$$

and this may be written

$$C = T = \frac{s}{c} \times \frac{1}{8}bd^2.$$

Therefore the resisting moment

$$M_r = \frac{s}{c} \times \frac{1}{8}bd^2 \times \frac{2}{3}d$$

$$= \frac{s}{c}\,\frac{bd^3}{12},$$

and hence,

$$M = \frac{s}{c}\,\frac{bd^3}{12} = \frac{sI}{c},$$

which is the same equation as used in the first method of solution. It is important to note that the resultants T and C do not act at a distance of $\frac{2}{3}$ of c from O unless the section of the beam is of constant width.

Problem 77.—A T-beam (Fig. 81) is subjected to a concentrated load of 4000 lb. at the center of the span. The beam is made of material having a tensile proportional limit of 4000 lb. per sq. in. and a compressive proportional limit of 8000 lb. per sq. in. Find the maximum tensile and compressive unit-stresses at the mid-span section, and find the ratios of these stresses to the proportional limits.

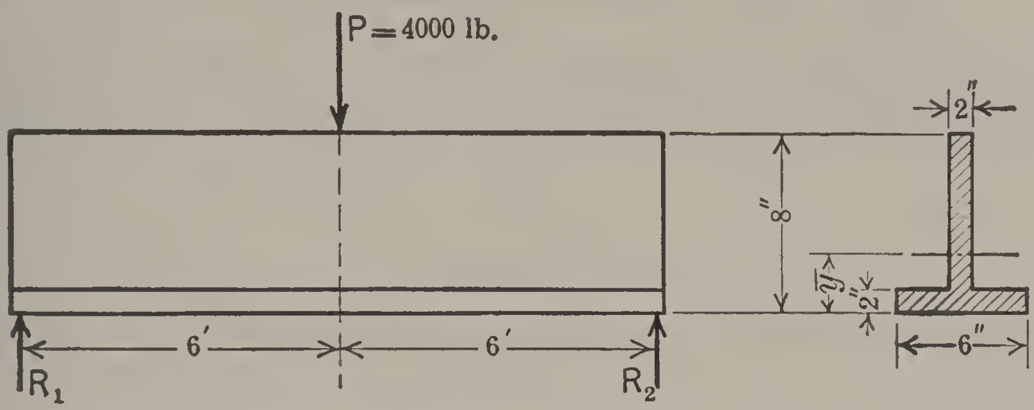


FIG. 81.—T-beam; used for material that is weak in tension.

Solution.—The distance, y, of the centroidal axis (and hence of the neutral axis) from the bottom line of the section is found to be 3 in. Thus (see Art. 160),

$$a\bar{y}=\Sigma a_0 y_0$$

$$\bar{y}=\frac{12\times5+12\times1}{12+12}=3 \text{ in.}$$

The moment of inertia of the cross-section with respect to the neutral axis is found as follows (see Art. 166):

$$\bar{I}_x=\tfrac{1}{12}2(6)^3+12\times(2)^2+\tfrac{1}{12}6(2)^3+12\times(2)^2$$

$$=136 \text{ in.}^4$$

The bending moment about the mid-section is

$$M=R_1\times\frac{l}{2}=2000\times72=144{,}000 \text{ lb.-in.}$$

Therefore, the maximum tensile unit-stress (on the bottom fiber of the beam) at the mid-section is

$$s_t=\frac{Mc}{I}=\frac{144{,}000\times3}{136}=3180 \text{ lb./sq. in.}$$

and the maximum compressive unit-stress (on the top fiber) is

$$s_c=\frac{5}{3}s_t=\frac{5}{3}\,3180=5300 \text{ lb. per sq. in.}$$

The ratio of each stress to the corresponding proportional limit is given below. Thus the tensile stress is about 0.8 of the tensile proportional limit whereas the compressive stress, although larger than the tensile stress, is only 0.66 of the compressive proportional limit of the material:

$$\frac{3180}{4000}=0.795 \text{ (tension)}, \quad \frac{5300}{8000}=0.662 \text{ (compression)}$$

This problem shows the advantage of a T-section (or a similar section) for material (such as cast iron) that is not equally strong in tension and compression.

PROBLEMS

78. Find the fiber unit-stress at a point on the section AB (Fig. 79) at a distance of 2 in. from the top face of the beam. *Ans.* $s=913$ lb. per sq. in.

79. Find the unit-stress on the bottom fiber of the beam in Fig. 79 on a section beneath the concentrated load.

80 Find the maximum compressive unit-stress on a section of the beam in Fig. 81 at a distance of 4 ft. from the left end.

81. A steel shaft 4 in. in diameter is used as a cantilever beam and loaded as shown in Fig. 82. Find the maximum unit-stress in the section, AB, at the wall. *Ans.* $s=6880$ lb. per sq. in.

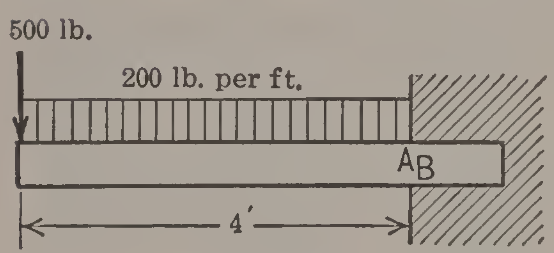


FIG. 82.

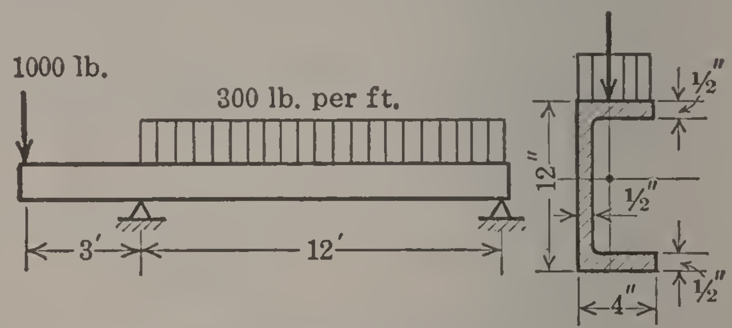


FIG. 83.

82. Find the maximum tensile fiber unit-stress on the section above the left support of the cast-iron beam shown in Fig. 83. Find also the maximum tensile fiber unit-stress on a section midway between the supports. State in each case whether the fiber on which the unit-stress occurs is on the top or bottom of the beam.

35. Section of Maximum Bending Moment.—If a beam has a constant cross-section throughout its length, the values of $\frac{I}{c}$ (section modulus) for all sections are equal, and hence the maximum value of s in the flexure formula, $M=\frac{sI}{c}$, occurs in the section at which the bending moment, M, is a maximum. It is important, therefore, to obtain a method of locating the section at which the bending moment is a maximum. This section is called the *dangerous section* of the beam, and may be located in accordance with the following statement:

> The section of a beam at which the bending moment is maximum is the section at which the vertical shear is either equal to zero or changes sign.

Proof.—In Fig. 84 is represented a beam subjected to a concentrated and a uniformly distributed load. The bending moment, M_x, at the distance x from the left support is

$$M_x = R_1 x - P(x-a) - \frac{wx^2}{2}.$$

Now the value of x that will make M_x a maximum is the value that will make the first derivative of M_x with respect to x equal to zero. The first derivative of M_x with respect to x is

$$\frac{dM_x}{dx} = R_1 - P - wx. \quad (40)$$

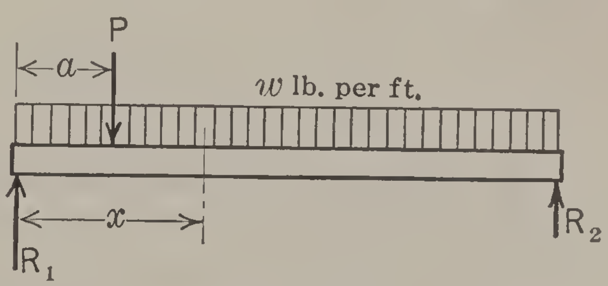


FIG. 84.

Therefore, the value of x that will make M_x a maximum may be found from the equation

$$R_1 - P - wx = 0.$$

But $R_1 - P - wx$ is the vertical shear for the section at the distance x from the left support. Therefore, the section at which the moment is maximum is the section for which the vertical shear is zero.

A convenient way of locating the section for which the vertical shear is zero (dangerous section) is to draw a shear diagram as discussed in the following article.

Further, the relation between the bending moment and the vertical shear found above, namely $V = \frac{dM}{dx}$, is of great importance; it will be discussed in greater detail in connection with shear and moment diagrams in the next article.

An alternative method of deriving this equation is as follows: In Fig. 85 is shown a part of a beam included between two sections a distance dx apart on which a distributed load acts; the load per unit of length being w. All the forces acting on this part of the beam are shown in Fig. 85, and since these forces hold the part in equilibrium, the algebraic sum of their moments about any point in the plane must equal zero. Hence if the point O is taken as the moment center we have,

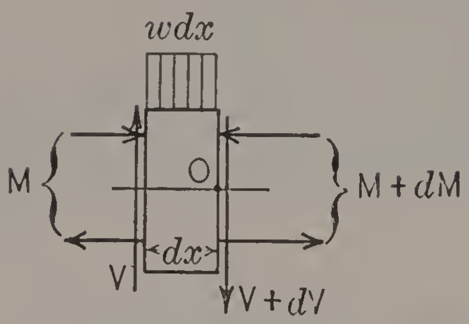


Fig. 85.

$$M+Vdx-wdx\times\tfrac{1}{2}dx-(M+dM)=0,$$

and since products of differentials may be neglected this equation reduces to

$$Vdx=dM \quad \text{or} \quad V=\frac{dM}{dx}. \qquad (41)$$

36. Shear and Moment Diagrams.—*Shear Diagram.*—A shear diagram for a beam is a curve in which the abscissas represent distances along the beam and the ordinates represent the vertical shears for the sections at which the ordinates are drawn.

For example let it be required to draw a shear diagram for the beam shown in Fig. 86(a). The reactions are found to be

$$R_1=7000 \text{ lb.} \quad \text{and} \quad R_2=5000 \text{ lb.}$$

Now the vertical shear for section A (just to the right of the left support) is $V_A=R_1=7000$ lb. upward; it is represented by AE in Fig. 86(b), an upward shear being considered positive and plotted above the base line.

The vertical shear for section C (just to the left of the load P) is

$$V_c=7000-4\times500=+5000 \text{ lb.}$$

and the vertical shear for any section between A and C at the distance x, say, from A is

$$V_x=7000-500x \qquad (42)$$

and hence the vertical shear decreases with x at the constant rate of 500 lb. per foot of length of the beam from 7000 lb. at A $(x=0)$ to 5000 lb. at C $(x=4)$ as shown in Fig. 86(b).

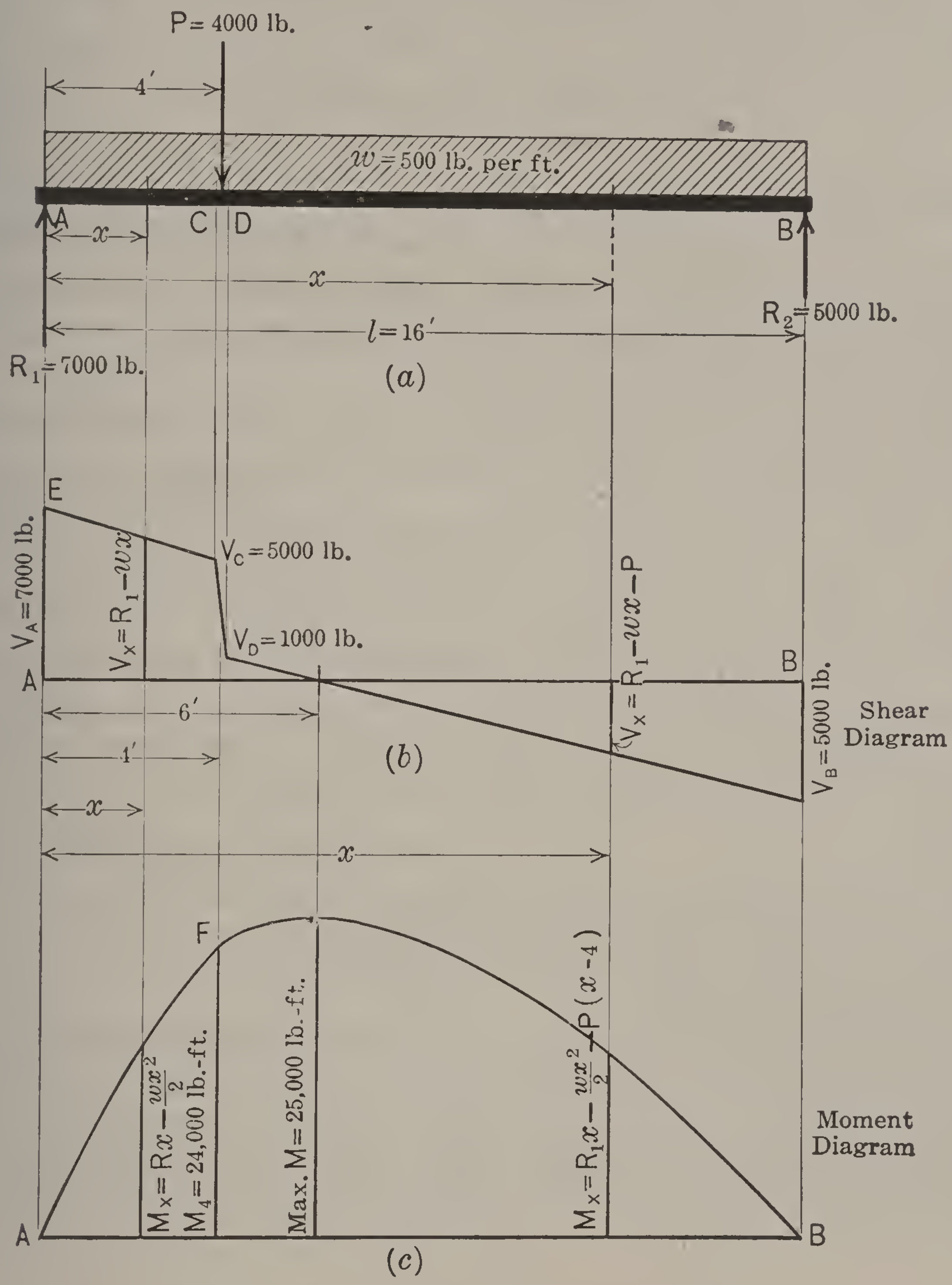


FIG. 86.—Shear and moment diagrams.

The vertical shear at section D (just to the right of the concentrated load) is

$$V_D = 7000 - 500 \times 4 - 4000 = +1000 \text{ lb.}$$

Thus, there is an abrupt change in the vertical shear in passing from sections C to D due to the concentrated load, but the shear does not (in this particular problem) pass through zero under the load; the shear must, therefore, be equal to zero for some section to the *right* of the concentrated load. Now, the vertical shear for any section between D and B at a distance x from the left support is

$$V_x = 7000 - 4000 - 500x, \quad \ldots \ldots \quad (43)$$

and hence the vertical shear decreases at the constant rate of 500 lb. per foot of length of the beam from 1000 lb. at D to -5000 lb. for a section just to the left of the right reaction. At 6 feet from the left support the vertical shear is zero and hence the bending moment at this section is maximum.

It is important to note that if a concentrated load acts on a beam in addition to a distributed load the shear may pass through zero under the load, and if it does the location of the dangerous section cannot be found by equating any expression such as equations (42) and (43) to zero and solving for x. It is desirable, therefore, in finding the dangerous section, to plot a shear diagram. If a beam is acted on by concentrated loads only, the dangerous sections will *always* occur under one of the loads since the shear changes only at the sections where the loads act.

Moment Diagram.—A *moment diagram* for a beam is a curve in which the abscissas represent distances along the beam and the ordinates represent the bending moments at the sections at which the ordinates are drawn.

Let it be required to draw a moment diagram for the beam shown in Fig. 86(a). The bending moment at the section A is equal to zero since the moment arm of R_1 about A is zero. The bending moment at any section between A and C, at a distance x from the left support, is

$$M_x = R_1 x - \frac{wx^2}{2}, \quad \ldots \ldots \ldots \ldots \quad (44)$$

in which x can not have a value greater than 4. The bending moment at any section between D and B (x greater than 4), at a distance x from the left support, is

$$M_x = R_1 x - \frac{wx^2}{2} - P(x-4), \quad \ldots \ldots \quad (45)$$

in which x cannot have a value less than 4. The value of the bending moment at the section beneath the load ($x=4$) may be found from either equation by making x equal to 4.

If various values of x from 0 to 4 be substituted in equation (44) and the resulting values of the bending moments plotted, the moment diagram obtained is AF (Fig. 86c), and if values of x from 4 to 16 be substituted in equation (45) the resulting moment diagram is FB (Fig. 86c).

It will be noted that the maximum ordinate to the moment curve occurs at the section for which the shear is zero. Further, it is important to observe that the moment diagram is composed of two *distinct* curves, AF and FB, which have only one point, F, in common, and that the maximum bending moment can be found from the equation of only one of these curves. Thus in the beam of Fig. 86(a) the value of x for which the bending moment is maximum can not be substituted in equation (44), since 4 is the greatest value x can have in this equation. If values of x greater than 4 are substituted in equation (44) the corresponding values of M_x will have no physical meaning; and if values of x less than 4 are substituted in equation (45) the corresponding values of M_x will have no physical meaning.

Sign of Bending Moment.—The bending moment at any section of a horizontal beam is considered to be positive when it produces compressive stress on the top fibers of the beam at the section and tensile stress on the bottom fibers. Thus, if the bending moment is obtained from the forces that lie to the left of the section the bending moment is positive when it is clockwise, and if obtained from the forces to the right of the section the bending moment is positive when it is counterclockwise.

ILLUSTRATIVE PROBLEMS

Problem 83.—A solid cylindrical steel shaft is used as a simple beam over a span of 12 ft. and is loaded as shown in Fig. 87(a). Draw the shear and moment diagrams and find the diameter of the shaft if the working fiber unit-stress is 16,000 lb. per sq. in.

Solution.—The reactions are found to be

$$R_1=6883 \text{ lb.} \quad \text{and} \quad R_2=5717 \text{ lb.}$$

The shear and moment diagrams are shown in Fig. 87(b). The dangerous section is under the concentrated load and the maximum moment is

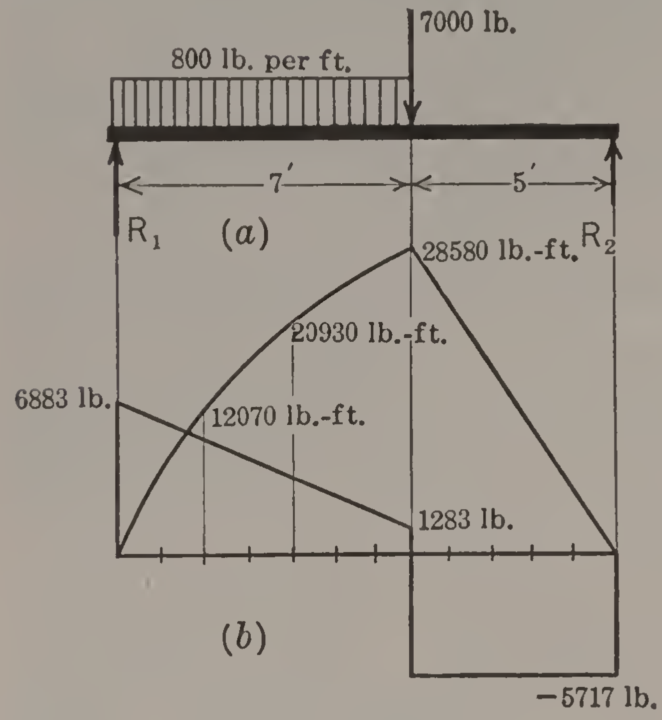


FIG. 87.

$$M = 6883 \times 7 - 800 \times 7 \times \tfrac{7}{2}$$
$$= 28{,}580 \text{ lb.-ft.}$$

The maximum fiber unit-stress is

$$s = \frac{Mc}{I} = \frac{28{,}580 \times 12 \times \frac{d}{2}}{\frac{\pi d^4}{64}}.$$

Hence,

$$d^3 = \frac{28{,}580 \times 12 \times 32}{16{,}000 \times \pi} = 218 \text{ in.}^3$$

whence

$$d = 6.02 \text{ in.}$$

Problem 84.—Show that the maximum bending moment for a simple beam loaded as shown in Fig. 88(a) is

$$M = \tfrac{1}{6}\, Wl$$

in which W is the total load on the beam and l is the span of the beam.

Solution.—From symmetry it is known that the shear is zero at the center of the span and that each reaction equals $W/2$. Let the intensity of the load at the center be w_c lb. per ft. Now the resultant, R, of the load to the left of the mid-span section is equal to the average load per foot ($w_c/2$) times the length (l). But the action line of R is at the distance of $\frac{2}{3} \times l/2$ from the left support. Thus the distributed load to the left of the center of the beam may be replaced by R as shown in Fig. 88(b) in getting the moment about the mid-span section. The bending moment about the mid-span section then, is

$$M = R_1 \times \frac{l}{2} - R\frac{1}{3}\frac{l}{2}$$
$$= \frac{1}{2}W\frac{l}{2} - \frac{w_c}{2}\frac{l}{2} \times \frac{1}{3}\frac{l}{2}$$

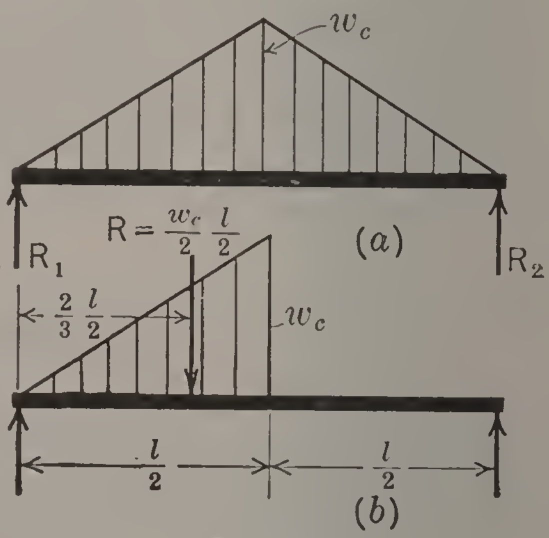


FIG. 88.

But, the average load per foot $\left(\frac{w_c}{2}\right)$ times the span length (l) equals W, $\left(W = \frac{w_c}{2}l\right)$.

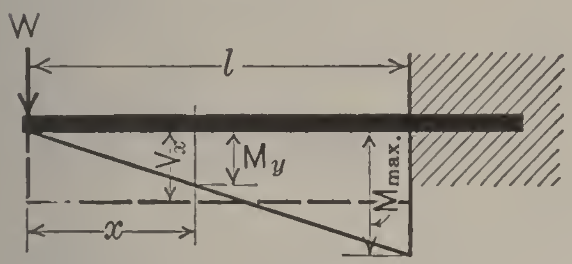
Hence,

$$M = W\frac{l}{4} - \frac{Wl}{12} = \frac{1}{6}Wl.$$

PROBLEMS FOR ARTICLES 34 TO 36

85 to **90.** Verify the shear and moment diagrams (shown in Figs. 89 to 94); also verify the expressions for the vertical shear and the bending moment at a section at the distance x from the left end of the bean, and the expressions for the maximum moment, as given in Figs. 89 to 94.

Prob. 85.

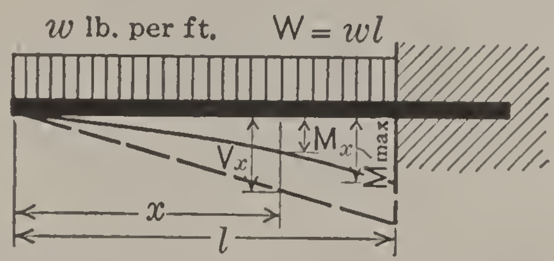


$$V_x = -W, \quad V_{\max.} = -W$$

$$M_x = -Wx, M_{\max.} = -Wl.$$

FIG. 89.

Prob. 86.

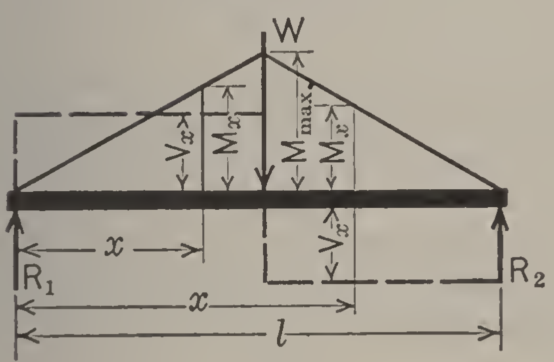


$$V_x = -wx, \quad V_{\max.} = -wl = -W.$$

$$M_x = -\frac{wx^2}{2}, \; M_{\max.} = -\frac{wl^2}{2} = -\frac{Wl}{2}.$$

FIG. 90.

Prob. 87.



$$V_x = \frac{W}{2}, \text{ when } x = 0 \text{ to } \frac{l}{2}.$$

$$V_x = \frac{W}{2} - W = -\frac{W}{2}, \text{ when}$$

$$x = \frac{l}{2} \text{ to } l.$$

$$V_{\max.} = \pm\frac{W}{2}.$$

$$M_x = \frac{W}{2}x, \text{ when } x = 0 \text{ to } \frac{l}{2}.$$

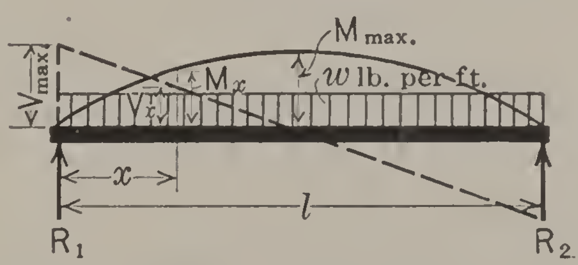
$$M_x = \frac{W}{2}x - W\left(x - \frac{l}{2}\right), \text{ when}$$

$$x = \frac{l}{2} \text{ to } l.$$

$$M_{\max.} = \frac{W}{2}\cdot\frac{l}{2} = \tfrac{1}{4}Wl.$$

FIG. 91.

Prob. 88.



$$V_x = \frac{W}{2} - wx \;, \; W = wl.$$

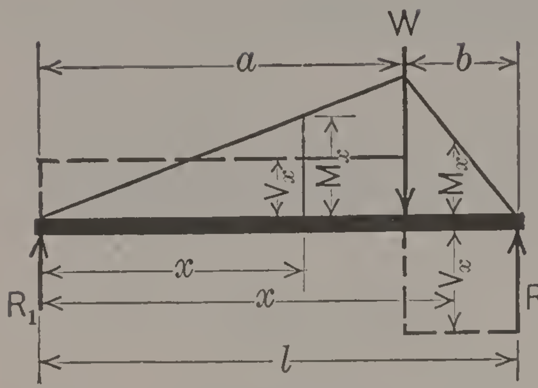
$$V_{\max.} = \pm\frac{W}{2}.$$

$$M_x = \frac{W}{2}x - \frac{wx^2}{2},$$

$$M_{\max.} = \frac{W}{2}\cdot\frac{l}{2} - \frac{W}{2}\frac{l}{4} = \tfrac{1}{8}Wl = \tfrac{1}{8}wl^2.$$

FIG. 92.

Prob. 89.



$$R_1=\frac{Wb}{l},\quad R_2=\frac{Wa}{l}.$$

$$V_x=\frac{Wb}{l} \text{ and } M_x=\frac{Wb}{l}x$$

when $x=0$ to a.

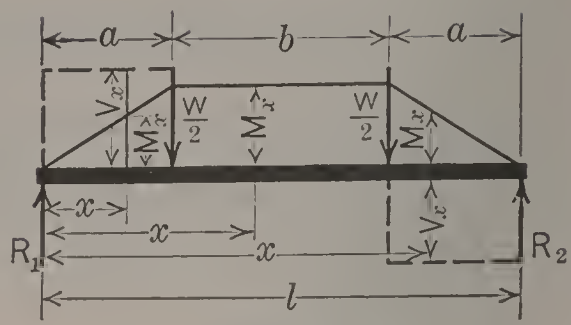
$$V_x=\frac{Wb}{l} \text{ and } M_x=\frac{Wb}{l}x-W(x-a)$$

when $x=a$ to l.

$$V_{\text{max.}}=-\frac{Wa}{l},\quad M_{\text{max.}}=\frac{Wb}{l}a.$$

FIG. 93.

Prob. 90.



$$V_x=\frac{W}{2}, \text{ and } M_x=\frac{W}{2}x$$

when $x=0$ to a.

$$V_x=0, \text{ and } M_x=\frac{W}{2}a,$$

when $x=a$ to $(a+b)$.

$$V_x=-\frac{W}{2}$$

and $$M_x=Wa-\frac{W}{2}(x-b)$$

when $x=(a+b)$ to l.

$$M_{\text{max.}}=\frac{W}{2}a.$$

FIG. 94.

91. Plot a shear diagram and find the dangerous section for the beam shown in Fig. 95. Find also the maximum bending moment and the maximum fiber unit-stress in the beam.

92. Draw to scale a shear and a moment diagram for the beam shown in Fig. 96. If the beam is a cylindrical bar what diameter should it have if the working fiber stress is 16,000 lb. per sq. in.? *Ans.* $d=4.97$ in.

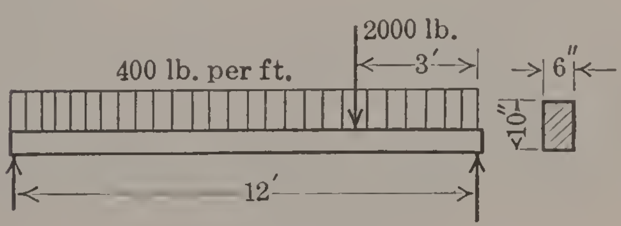


FIG. 95.

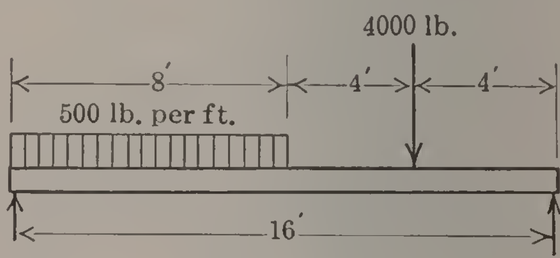


FIG. 96.

93. A simple beam having a span of 12 ft. is made of yellow pine for which the working fiber stress is 1000 lb. per sq. in. The beam carries concentrated loads of 2000, 4000 and 6000 lb. at distances of 3, 6 and 7 ft., respectively, from the left support and also a total uniformly distributed load of 1200 lb. If the width of the beam is 10 in. what should be the depth?

Ans. $d=15.2$ in.

94. A freight-car axle (Fig. 97) is 4 in. in diameter. The distance from the center of an axle box to the action line of the rail pressure is 8 in. and the distance between the rail pressures is 4.90 ft. Find the maximum fiber unit-stress in the axle if each rail pressure is 5 tons.

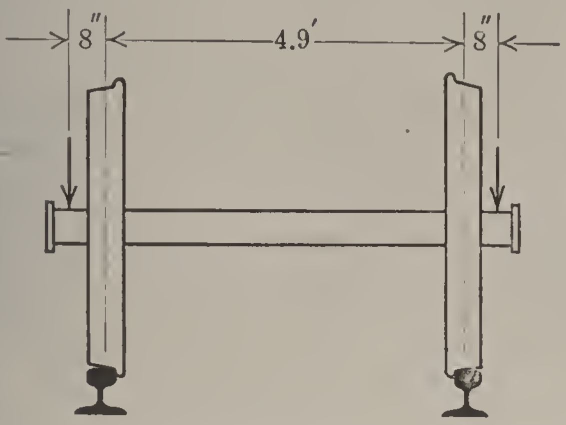


FIG. 97.

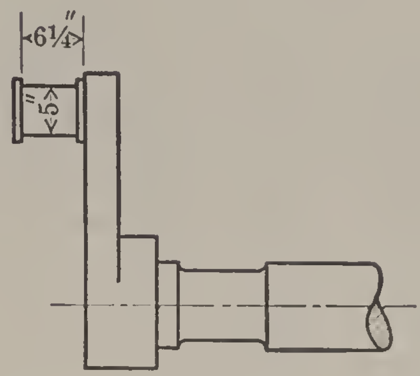


FIG. 98.

95. The steel crankpin (Fig. 98) of a steam engine has a force of 25,000 lb. transmitted to it by the connecting rod. The dimensions of the pin as shown in Fig. 32 were determined from the allowable bearing pressure. Find the allowable bearing pressure, and the fiber unit-stress. The load may be assumed to be uniformly distributed over the length of the pin.

Ans. $s_b = 800$ lb. per sq. in., $s = 6360$ lb. per sq. in.

96. A simple beam having a span of 16 ft. carries a total uniformly distributed load of W lb., and a concentrated load, equal to $W/2$, at the center of the span. If the cross-section of the beam is rectangular, 6 in. wide by 12 in. deep, and the allowable fiber unit-stress is 800 lb. per sq. in., find the value of W.

97. The cast-iron frame shown in Fig. 99 is subjected to a load P of 4000 lb. Find the maximum tensile and compressive fiber stresses at the section AB.

Ans. $\bar{y}_1 = 3.5$ in., $I = 392$ in.^{4}, $s_t = 857$ lb. per sq. in., $s_c = 1590$ lb. per sq. in.

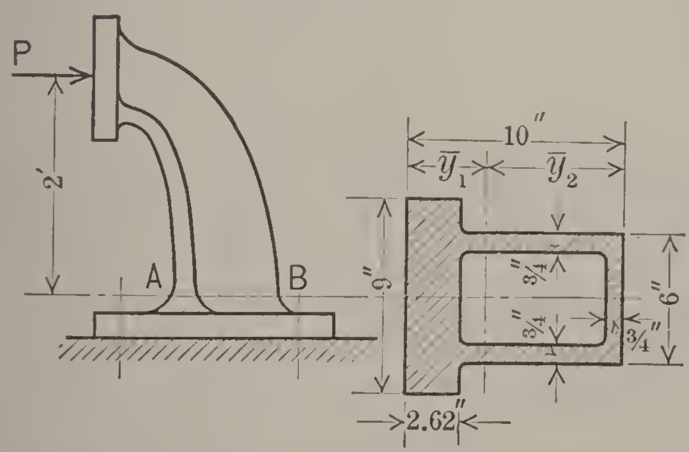


FIG. 99.

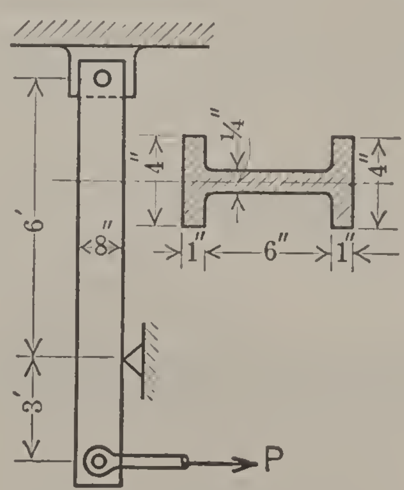


FIG. 100.

98. What force P (Fig. 100) will cause a maximum fiber unit-stress of 16,000 lb. per sq. in. in the I-beam ?

99. The load P (Fig. 101) is 1000 lb. What should be the depth d of the timber beam A if the working unit-stress in the material is 1200 lb. per sq. in ?

Ans. $d = 9.5$ in.

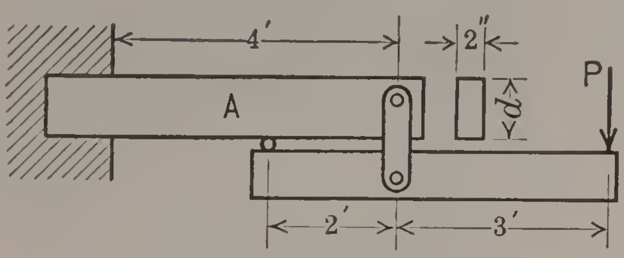


FIG. 101.

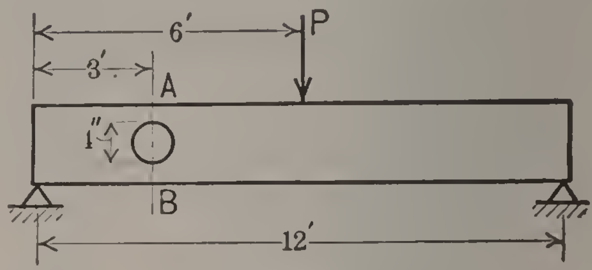


FIG. 102.

100. The simple timber beam shown in Fig. 102 has a hole 4 in. in diameter bored through the beam, the center line of the hole lying in the neutral surface and in the section AB, 3 ft. from the left support. The beam is subjected to a load P of 5000 lb. Find the maximum unit-stress that occurs in the section AB and also in the section for which the vertical shear is zero. The beam has a rectangular section 4 in. wide and 12 in. deep.

101. Sand is piled on a floor so that the load on the beams supporting the floor varies from zero pressure at one end to a pressure of w lb. per ft. at the other end (Fig. 103). Show that the vertical shear is zero for a section at a distance of $0.59l$ from the left support, and that the maximum bending moment is

FIG. 103.—Non-uniformly distributed load.

$$M = \frac{104}{810} Wl. = 0.128\,Wl \quad . \quad . \quad (47)$$

102. Draw the shear and moment diagrams for the beam shown in Fig. 104. Find the maximum unit-stress in the beam if the beam has a rectangular cross-section 4 in. wide by 8 in. deep.

Ans. $s = 983$ lb. per sq. in.

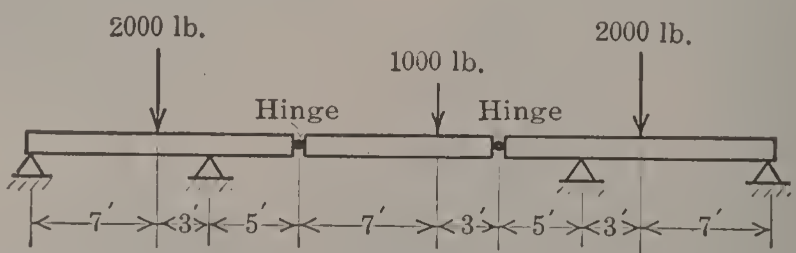


FIG. 104.

NOTE.—The following problems are to be solved with the aid of a steel company's handbook. Find the missing terms in the table. All the beams are simple beams and all distances are measured from the left support.

Prob. No.	Length of Span (feet)	Uniform Load (lb./ft.)	Concentrated Load (lb.)	Distance to Concentrated Load (ft.)	I-Beam Size
103	10	2000	0	0	?
104	12	?	0	0	10″ 25-lb.
105	14	0	28000	7	?
106	14	0	?	7	9″ 21-lb.
107	16	1000	10000	8	?

37. Relation between Shear and Moment.—In Art. 35 it was shown that

$$V=\frac{dM}{dx},$$

which states that the vertical shear for a section of a beam is equal to the rate, with respect to the distance along the beam, at which the bending moment is changing at the section. Or, the equation may be written

$$dM=Vdx,$$

which states that the difference, dM (Fig. 105), between the bending moments at two sections that are the distance dx apart is equal to the area, Vdx, under the shear curve between the two sections. Now the difference (M_2-M_1 or ΔM) between the bending moments at the sections x_2 and x_1 would be

$$\Delta M=\int_{M_1}^{M_2} dM$$

(Fig. 105). But, from the above equation,

$$\int_{M_1}^{M_2} dM=\int_{x_1}^{x_2} Vdx. \quad . \quad . \quad (48)$$

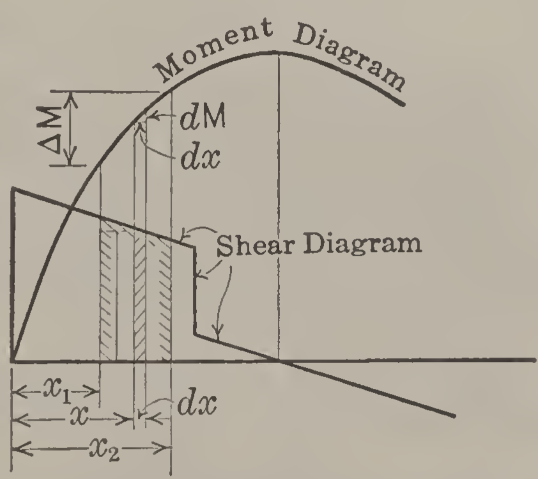


FIG. 105.—Relation between bending moment and vertical shear.

And $\int_{x_1}^{x_2} Vdx$ represents the area under the shear curve between the ordinates x_1 and x_2, and hence,

$M_2-M_1=$ area under the shear curve between ordinates x_1 and x_2. Therefore, the difference between the bending moments at two sections of a beam is represented by the area under the shear curve between ordinates at the two sections.

Thus, the bending moment at a given section of a simple beam is represented by the area under the shear curve between the end of the beam ($M=0$) and the given section. For example, let it be required to find the maximum moment for a simple beam subjected to a total load W, uniformly distributed over the entire span, l. The shear diagram is shown in Fig. 106. According to the above proposition the difference between the bending moments at the center and end is represented by the area under the shear curve between these two sections. But the moment at the end is zero and hence the moment at the center is

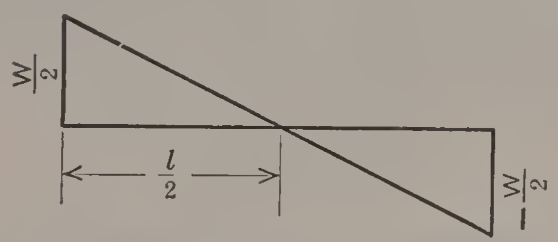


FIG. 106.—Shear diagram.

$$M_c = \tfrac{1}{2}\ \text{base} \times \text{altitude}$$

$$= \frac{1}{2}\frac{W}{2} \times \frac{l}{2} = \frac{l}{8}Wl,$$

which checks the result found in Prob. 88.

PROBLEMS

108. Show, by the above method, that the maximum bending moment for a simple beam carrying a concentrated load P at mid-span is, $M = \frac{1}{4}Pl$.

109. Show, by the above method, that the bending moment at any section within the middle third of a beam when loaded with equal concentrated loads at the third-points is, $M = \frac{1}{3}Pl$.

110. Find, by the above method, the maximum moment for the beam shown in Fig. 93.

38. Overhanging Beams.—The discussion in the preceding articles concerning the stresses developed in simple and cantilever beams also applies to overhanging beams. The bending moment at a section in an overhanging beam, however, may be negative, and there are two maximum moments to be considered—the maximum positive moment and the maximum negative moment. In other words, the vertical shear passes through zero at two or more sections of the beam. Further, since the moment changes from a positive to a negative value it must be zero at some section; the section at which the bending moment is zero is called the *point of inflection.*

The above facts will be discussed in connection with the following illustrative problem.

ILLUSTRATIVE PROBLEM

Problem 111.—A 6-in. by 12-in. by 14-ft. timber beam is supported and loaded as shown in Fig. 107(a). (a) Draw the shear and moment diagrams. (b) Find the position of the dangerous section. (c) Find the maximum positive moment. (d) Find the maximum negative moment. (e) Find the point of inflection. (f) Find the maximum fiber unit-stress.

Solution.—The reactions are found to be

$$R_1 = 4600 \text{ lb.} \quad \text{and} \quad R_2 = 7400 \text{ lb.}$$

(a) The shear and moment diagrams are shown in Fig. 107(b). The vertical shear is equal to zero for a section 3.25 ft. from the left support, and passes through zero at the section above the right support. These two sections, then, are the dangerous sections. The bending moment at any section to the left of the first concentrated load (P_1), at a distance x from the left support, is

$$M_x = 4600x - 800\frac{x^2}{2}, \quad . \quad (49)$$

in which x has any value from 0 to 3. The bending moment at any section between P_1 and the right reaction is

$$M_x = 4600x - 800\frac{x^2}{2} - 2000(x-3), \quad (50)$$

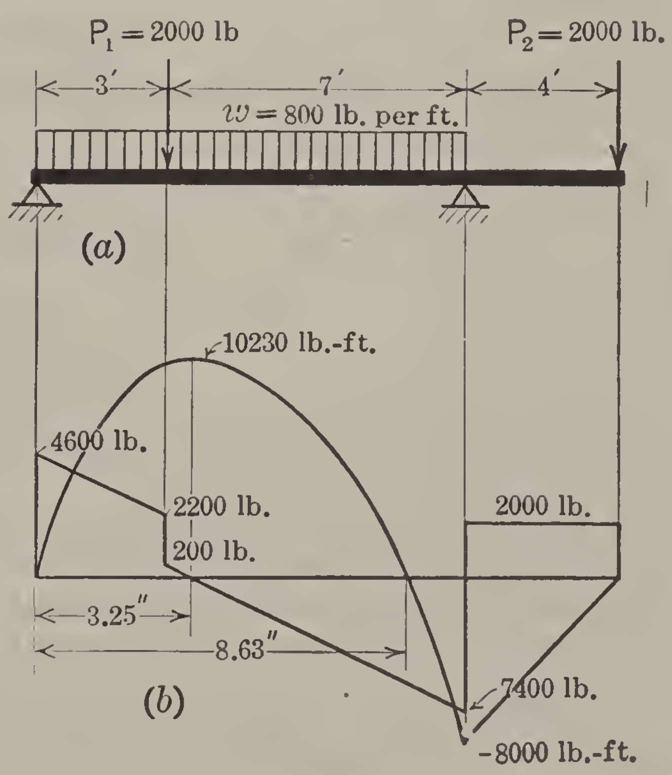


FIG. 107.

in which x cannot have a value less than 3 ft. or greater than 10 ft. The bending moment at any section between the right reaction and the right end of the beam is

$$M_x = 4600x + 7400(x-10) - 2000(x-3) - 8000(x-5), \quad . \quad . \quad . \quad (51)$$

in which x cannot be less than 10 or greater than 14 ft. Or, if x is measured from the right end of the beam instead of the left end then the bending moment at any section between R_2 and P_2 is

$$M_x = -2000x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Equations (51) and (52) will give, of course, the same value for the bending moment at a given section of the beam.

(*b*) The maximum positive moment is

$$M_{\text{pos.}} = 4600 \times 3.25 - 2000 \times 0.25 - \frac{800 \times (3.25)^2}{2}$$
$$= 10{,}230 \text{ lb.-ft.}$$

(*c*) The maximum negative moment, obtained from the forces to the left of the section, is

$$M_{\text{neg.}} = 4600 \times 10 - 2000 \times 7 - 800 \times 10 \times 5$$
$$= -8000 \text{ lb.-ft.}$$

Or, when obtained from the forces to the right of the section, it is

$$M_{\text{neg.}} = -2000 \times 4 = -8000 \text{ lb.-ft.}$$

(*d*) The bending moment changes signs at a section between the dangerous sections. Hence, to locate the inflection point equation (50) may be equated to zero. Thus,

$$M_x = 4600x - 800\frac{x^2}{2} - 2000(x-3) = 0.$$

Therefore,

$$x = 0 \text{ or } 8.63 \text{ ft.}$$

and hence the inflection point is 8.63 ft. from the left support.

(*e*) Since the maximum positive moment is larger than the maximum negative moment, the maximum fiber unit-stress is

$$s = \frac{Mc}{I} = \frac{10230 \times 12 \times 6}{\frac{1}{12}6 \times (12)^3}$$
$$= 852 \text{ lb. per sq. in.}$$

PROBLEMS

For each of the beams described in the following problems obtain: (*a*) The shear and moment diagrams; (*b*) the maximum positive and negative bending moments; (*c*) the points of inflection; (*d*) the maximum fiber unit-stress in the beam. The weight of the beam may be neglected in each problem.

112. The beam shown in Fig. 108 has the following dimensions: $b = 8$ in., $d = 10$ in., and the loads are, $P = 1000$ lb. and $w = 300$ lb. per ft.

Ans. (*d*) $s = 1050$ lb. per sq. in.

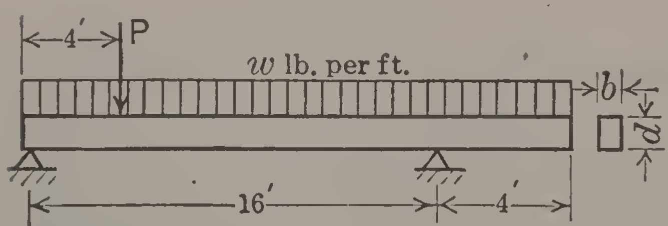


Fig. 108.

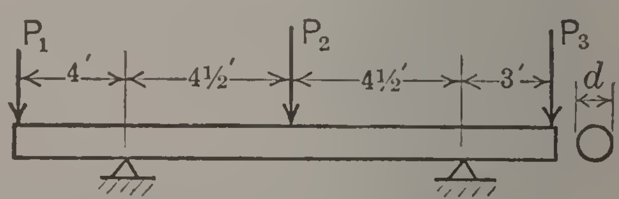


Fig. 109.

113. The diameter, d, of the beam shown in Fig. 109, is 4 in. and the loads are $P_1 = 1500$, $P_2 = 6000$ lb., and $P_3 = 3000$ lb.

114. See the beam shown in Fig. 73, and consider the beam to have a rectangular cross-section 4 in. wide and 12 in. deep.

39. Economical Sections of Beams.—The efficient use of material in a force-resisting member requires (*a*) the selection of available material that is well suited to the use of the member, and (*b*) the distribution of the material in the member so that it can best resist the forces acting on the member. The use of material, however, must always be considered in relation to the cost involved.

The selection of engineering materials best suited to various purposes is discussed in Part II. We are here interested mainly in the effect of the distribution of the material in a beam on the resistance of the beam to external loads; this involves (*a*) the effect of the shape of cross-section and (*b*) the effect of a variation of size of cross-section along the beam.

(*a*) *Effect of Shape of Cross-section.*—If a beam having a constant cross-section safely resists static loads the maximum fiber

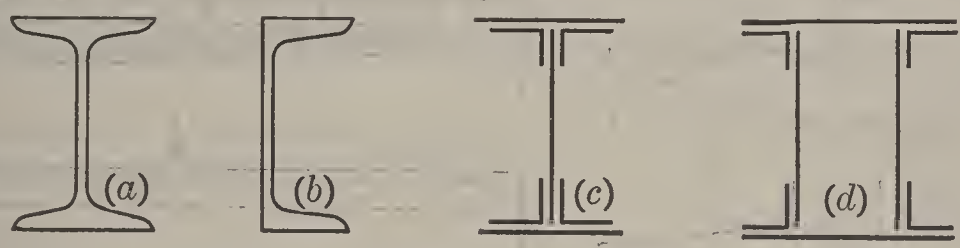


FIG. 110.—Sections having large values of $\frac{I}{c}$.

unit-stress at the dangerous section must not exceed the allowable or working unit-stress that experiments and experience have shown to be permissible. That is, s in the flexure formula, $s=\frac{Mc}{I}$, must not be greater than the working unit-stress when M is the maximum bending moment.

The flexure formula shows that, when a given unit-stress s is developed in the beam, the bending moment M required to develop this unit-stress is large when $\frac{I}{c}$ is large. Now $\frac{I}{c}$ is made large by forming the cross-section so that the greater part of the area is as far from the neutral axis as practicable. Thus, steel beams are rolled in the form of I-sections (Fig. 110*a*), channel-sections (Fig. 110*b*), etc.; further, built-up steel beams of various shapes (Fig. 110*c* and *d*), and also cast-iron beams are made to conform to this principle. Steel beams are made with the two flanges equal in area since the proportional limits in compression and tension are approximately equal. Cast-iron beams, however, are cast with the tensile flanges larger in area than the compressive flange (see Fig. 99) since the compressive strength of cast iron is much larger (about four times) than the tensile strength. Although cast-iron beams are rarely used in buildings or bridges, they are frequently used in machine frames that are subjected to bending.

(b) *Effect of a Varying Section Along the Beam.*—As discussed in Art. 36 the bending moment, in general, varies along a beam and is maximum at one section of the beam. If, then, a beam has a constant cross-section $\left(\frac{I}{c} = \text{a constant}\right)$ the maximum fiber unit-stress will occur on the outer fiber of the section at which the

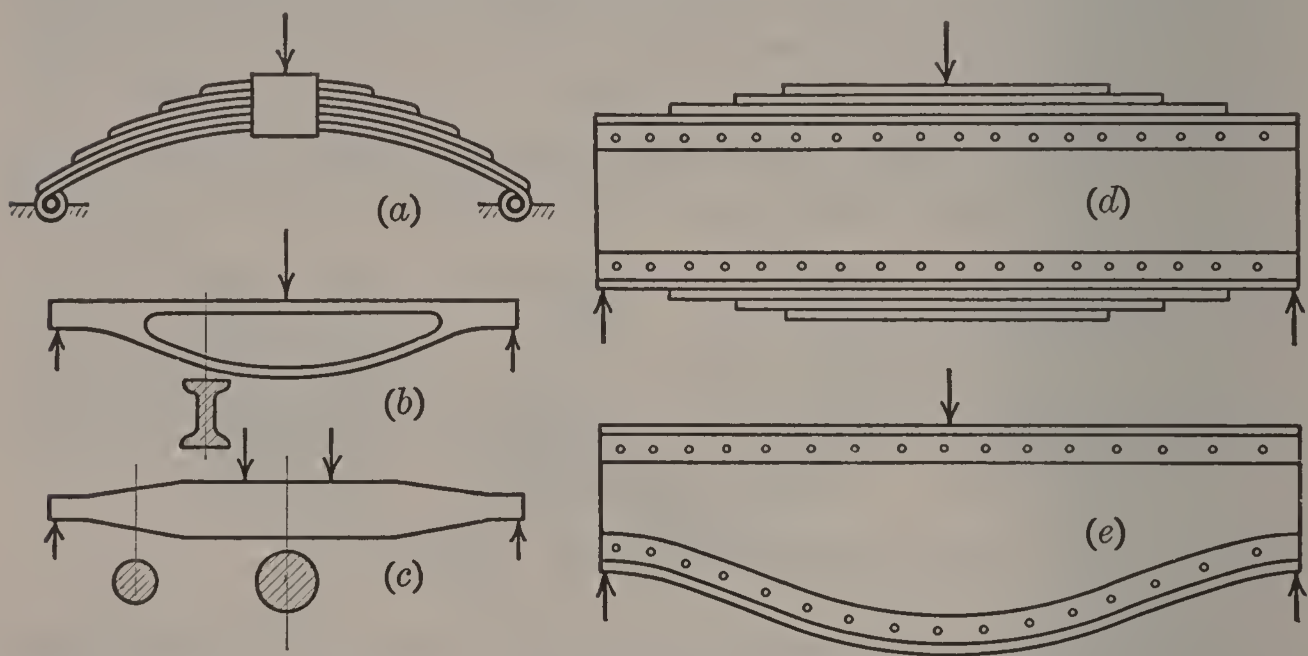


FIG. 111.—Beams that approach the conditions for uniform stress.

bending moment is maximum, and the unit-stress in the outer fibers at all other sections will be less than that at the dangerous section. Therefore, when the beam is carrying the load that causes the allowable fiber unit-stress in the beam there is much material in the beam on either side of the dangerous section that is under-stressed, and hence this under-stressed material could be saved by varying the cross-section of the beam so that the $\frac{I}{c}$ would vary as the bending moment M varies. This would cause the unit-stresses in the outer fibers of all sections to be equal, since $s = \frac{M}{\frac{I}{c}}$.

Rolled steel beams nearly always are beams of constant cross-section since the cost of rolling a beam of variable cross-section would offset the saving in material. Built-up beams, such as leaf springs (Fig. 111*a*) and plate girders (Fig. 111*d*), frames of cars (Fig. 111*e*), forged axles (Fig. 111*b*) and turned axles (Fig. 111*c*), etc., are frequently made with a variable section so that the beam approximates a beam of uniform strength.

40. Shearing Stress in a Beam.—In Art. 33 it was shown that in a section of a beam there is shearing stress as well as tensile and compressive stresses, and that the total resisting shearing stress, V_r, in any section is equal to the vertical shear, V, for the section. That is, $V = V_r$. In the preceding articles the tensile (or compressive) unit-stress at any point in the beam was found in terms of the external loads and the dimensions of the beam, and the problem now to be considered is that of expressing the shearing unit-stress at any point in a beam in terms of the external loads and the dimensions of the beam.

If the shearing unit-stress on the cross-sectional area, a, of a beam were constant and equal to s_s the resisting shear, V_r, would be equal to as_s, and hence the shearing unit-stress at any point of the area would be equal to $\frac{V}{a}\left(s_s = \frac{V}{a}\right)$. But, the shearing unit-stress is not constant over the area, as will be shown later in this article, and hence the equation $s_s = \frac{V}{a}$ gives *only the average* shearing unit-stress. Shearing stress in beams is of importance mainly in timber beams, concrete beams, and some built-up steel beams.

As stated in Art. 17, if a shearing unit-stress occurs on one plane at a point in a body, a shearing unit-stress of equal magnitude must occur at the same point on another plane at right angles to the first plane. Thus, in Fig. 112, X represents a small block in a beam; there is a vertical upward shearing stress on its left face and a vertical downward shearing stress on its right face, but these two forces form a couple which would rotate the block and hence, since the block is in equilibrium, there must be horizontal shearing forces on its upper and lower faces as indicated in Fig. 112. Thus we are led to the conclusion (see Art. 17 for proof) that the vertical and horizontal shearing unit-stresses *at any point* are equal.

The horizontal shearing unit-stress (and also the vertical shearing unit-stress) at any point in a beam may be found, in terms of the external forces and the dimensions of the beam as follows:

Let a block, B, having a small width dx (Fig. 112) be removed from the beam and be replaced by the forces that the block was exerting on the beam. These forces must have been exerted, of course, on the faces that were in contact with the beam. Now, as shown in Art. 34, the compressive (or tensile) unit-stress at any point on any cross-section of the beam varies directly as the distance of the point from the neutral axis, being zero at the neutral axis. Thus the block B must have pushed horizontally at its two end faces as shown in Fig. 112(a) (or the beam must-have pushed on the end faces of the block as shown in Fig. 112b).

But since the bending moments at the two sections are not, in general, equal, the sum of the stresses on the two faces are not equal and hence there must be a shearing stress on the bottom face of the block.

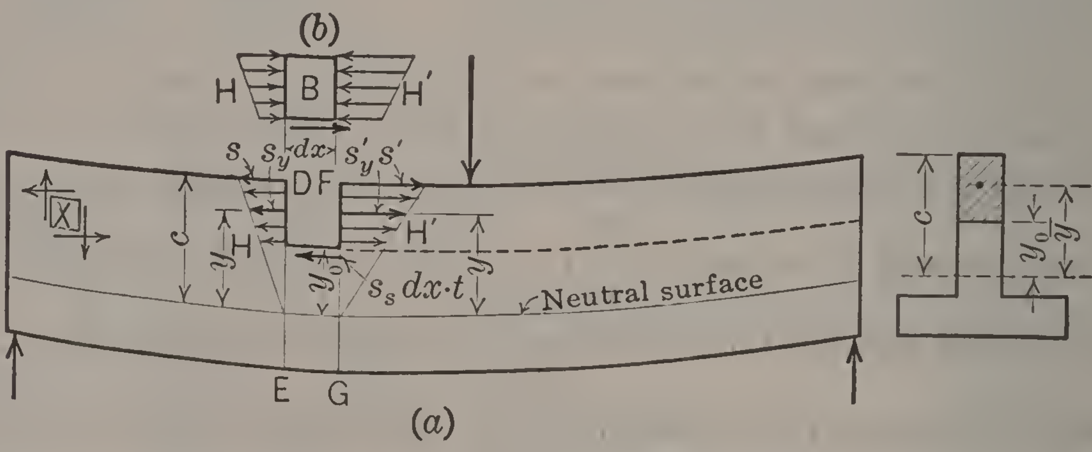


FIG. 112.—Shearing stress in a beam.

Thus, if H' and H represent the sum of the stresses on the right and left faces, respectively, the total shearing stress on the bottom force is equal to $H'-H$, assuming H' to be larger than H. Further, since the area $dx \cdot t$ on which the shearing stress occurs is small the shearing unit-stress, s_s, may be assumed to be constant over the area and hence the total shearing stress is $s_s \cdot dx \cdot t$. Therefore,

$$s_s dx \cdot t = H' - H.$$

Now $H' = \int_{y_0}^{c} s_y' da$ in which s'_y is the unit-stress in the plane FG on a fiber at the distance y from the neutral axis (Fig. 112a). Further, $\frac{s_y'}{y} = \frac{s'}{c} =$ a constant (Art. 34.) Therefore,

$$H' = \frac{s'}{c}\int_{y_0}^{c} y da \text{ and similarly } H = \frac{s}{c}\int_{y_0}^{c} y da.$$

Hence

$$s_s dx \cdot t = \frac{s'-s}{c}\int_{y_0}^{c} y da.$$

But $s' = \frac{M'c}{I}$ and $s = \frac{Mc}{I}$ in which M' and M are the bending moments at the right and left sections respectively. Therefore,

$$s_s dx \cdot t = \frac{M'-M}{I}\int_{y_0}^{c} y da.$$

But $M' - M = dM$ since the two sections are the small distance dx apart. Hence

$$s_s = \frac{dM}{dx} \cdot \frac{1}{It}\int_{y_0}^{c} y da,$$

and from Art. 37,

$$\frac{dM}{dx} = V.$$

Therefore,

$$s_s = \frac{V}{It}\int_{y_0}^{c} y da, \quad \text{. . . . . (53)}$$

in which s_s is the horizontal shearing unit-stress (and also the vertical shearing unit-stress) in a cross-section for which the vetical shear is V, and at a point whose distance from the neutral axis is y_0, the thickness of the beam at the distance y_0 from the neutral axis being t; I is the moment of inertia of the whole cross-section of the beam about the neutral axis. The expression $\int_{y_0}^{c} y da$ is the first moment (often called the statical moment), about the neutral axis, of that part of the cross-sectional area of the beam between the plane on which the horizontal shearing unit-stress s_s occurs and the outer face of the beam (that is, between y_0 and c). This area is the cross-hatched area a' in the end view in Fig. 112a. Further, if the distance, $\bar{y}$, of the centroid of the area a' from the neutral axis is known, the moment of this area may be found from the product $a'\bar{y}$ since

$$\int_{y_0}^{c} y da = a'\bar{y} \quad \text{(Art. 158).}$$

Hence the above equation may be written

$$s_s = \frac{V}{It} \cdot a'\bar{y}. \quad \text{. . . . . (54)}$$

The above equations are valid only when the tensile or compressive unit-stress in the beam does not exceed the proportional limit of the material since the flexure formula, which is based on this assumption, was used in its derivation.

It is important, next, to locate the point in a beam at which the shearing unit-stress, s_s, is a maximum. If the beam has a constant cross-section, I and t are constant, and hence the maximum value of s_s in equations (53) and (54) will occur in the section of the beam for which V is maximum; for a *simple* beam subjected to a uniform load V is maximum close to one support. Further, in any section of constant thickness, s_s will be maximum when $\int_{y_0}^{c} y\,da$ or $a'\bar{y}$ is

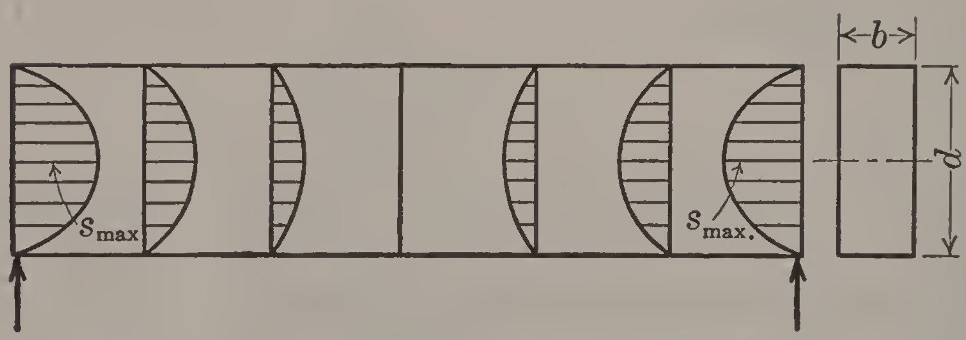


FIG. 113.—Variation of shearing stress in length of beam.

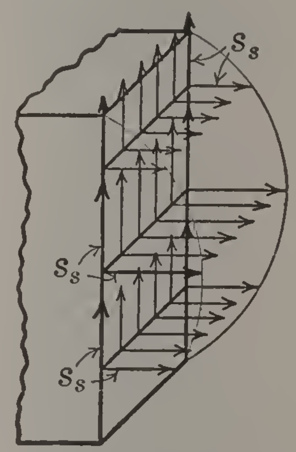


FIG. 114.—Variation of shearing stress over section of beam.

maximum, which occurs when y_0 is zero, that is, s_s is maximum at the neutral surface. Thus in a *simple* beam of *constant thickness*, subjected to a uniform load the horizontal shearing unit-stress varies throughout the beam as shown in Fig. 113, and at any point in a section the horizontal and vertical shearing unit-stresses are equal as indicated in Fig. 114.

Thus, the maximum shearing unit-stress in any section of a beam having a *rectangular* cross-section bd (Fig. 113) is

$$s_s = \frac{V}{It} a_1 \bar{y} = \frac{V}{\frac{1}{12}bd^3 \cdot b} \cdot b\frac{d}{2} \times \frac{d}{4}$$

$$= \frac{3}{2}\frac{V}{bd} = \frac{3}{2}\frac{V}{a},$$

in which a is the area of the whole cross-section. Hence the maximum value s_s in any section of a *rectangular* beam is 50 per cent greater than the average shearing unit-stress in the section. If a beam has a *circular* cross-section the maximum value of s_s is $33\frac{1}{3}$ per cent greater than $\frac{V}{a}$, that is,

$$s_s = \frac{4}{3}\frac{V}{a}.$$

ILLUSTRATIVE PROBLEMS

Problem 115.—Find the maximum shearing unit-stress in a 15-in. 45-lb. channel (Fig. 115*a*) when used as a simple beam on a 12-ft. span and subjected to a concentrated load of 20,000 lb. at the center of the span. (Neglect the weight of the beam.)

Solution.—A steel maker's handbook gives the following values in addition to those shown in Fig. 115(*a*).

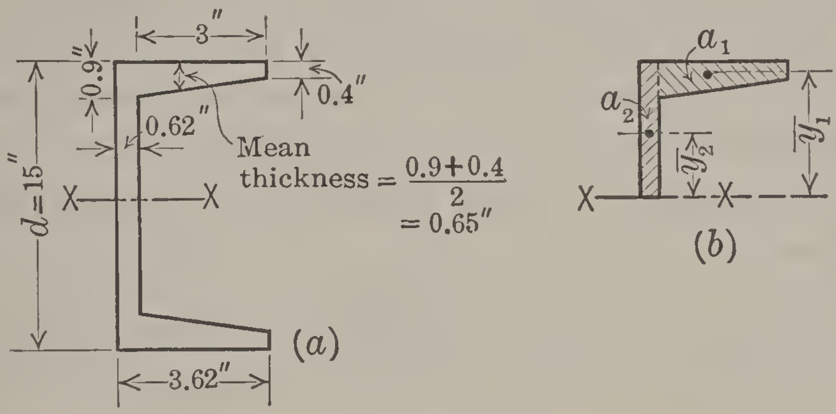


FIG. 115.—15-in. 45-lb. channel section.

$$I_x = 375 \text{ in.}^4 \quad a = 13.2 \text{ sq. in.}$$

The maximum shearing unit-stress is

$$s_s = \frac{V}{It}a'\bar{y}$$

$$V = 10{,}000 \text{ lb.} \quad I = 375 \text{ in}^4. \quad t = 0.62 \text{ in.}$$

The value of $a'\bar{y}$ is most easily found as the sum of the moments of the areas a_1 and a_2 (Fig. 115*b*). Thus,

$$\begin{aligned} a'\bar{y} &= a_1\bar{y}_1 + a_2\bar{y}_2 \\ &= (3\times 0.65)\times\left(7.5 - \frac{0.65}{2}\right) + (0.62\times 7.5)\times 3.75 \\ &= 31.5 \text{ in}^3. \end{aligned}$$

Therefore

$$\begin{aligned} s_s &= \frac{10{,}000}{375}\cdot\frac{31.5}{0.62} \\ &= 1350 \text{ lb. per sq. in.,} \end{aligned}$$

and hence the maximum shearing unit-stress in the beam is 1350 lb. per sq. in., provided that the tensile or compressive unit-stress does not exceed the elastic limit of the material.

The maximum tensile or compressive unit-stress is

$$s=\frac{Mc}{I}=\frac{10{,}000\times 72\times 7.5}{375}=14{,}400 \text{ lb. per sq. in.}$$

which is less than the elastic limit of the material.

The average shearing unit-stress is

$$s_{\text{av.}}=\frac{V}{a}=\frac{10{,}000}{13.24}=755 \text{ lb. per sq. in.}$$

and hence the maximum shearing unit-stress is 1.79 times the average.

Shearing Stress in Web of I-beam.—It is customary in structural design to assume that the flanges of a channel-beam or an I-beam are effective in resisting bending but not in resisting shear, and the maximum shearing unit-stress in the beam is assumed to be equal to the average shearing unit-stress over the web only, regarded as extending the entire depth of the beam. Thus in the above problem

$$(s_s)_{\text{av.}} \text{ in web}=\frac{V}{td}=\frac{10{,}000}{0.62\times 15}=1074 \text{ lb. per sq. in.}$$

This method, therefore, gives only approximate results for the channel section, although in the case of an I-section the method yields results somewhat closer to those found by the correct method.

The justification for assuming the shearing unit-stress in an I-beam to be distributed uniformly over the web considered to extend the entire depth of the beam may be found in a study of Fig. 116. In any section of an I-beam s_s, according to equation (54), will be large where t is small, and small where t is large, the effect on s_s of the term $\int_{y_0}^{c} y\,da$ being relatively small in the case of an I-section; further, s_s will change abruptly where t changes abruptly. Hence, the horizontal (and vertical) shearing unit stress at a section in an I-beam varies as shown in Fig. 116(b), and

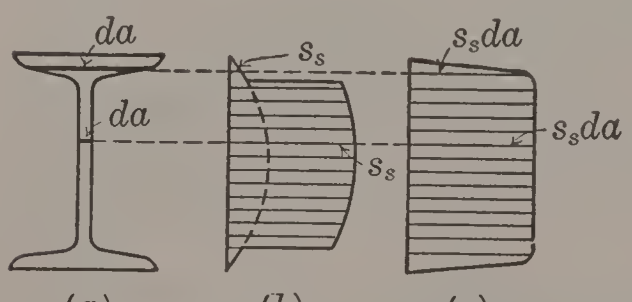


FIG. 116.—Distribution of shearing stress over section of I-beam.

the total shearing stress ($s_s \cdot da$) on the elementary strips of the section (the total stress per unit of depth) is approximately constant for the whole depth of the section as shown in Fig. 116(c). Therefore, the method, stated above, for finding the shearing unit-stress in the web of an I-beam yields results that contain small errors.

PROBLEMS

116. A simple beam made of wood has a cross-section 6 in. wide by 8 in. deep. A concentrated load of 16,000 lb. acts at the mid-span section, and the span is 10 ft. (a) Find the maximum shearing unit-stress. (b) Find also the shearing unit-stress at a point 2 in. from the top of the beam in a section 2 ft. from the left support.

Ans. (a) $s_s = 250$ lb. per sq. in. (b) $s_s = 188$ lb. per sq. in.

117. A simple timber beam has a hollow square cross-section; the outside dimensions are 8 in. by 8 in. and the inside dimensions are 4 in. by 4 in. The span of the beam is 10 ft. and the beam is subjected to a uniformly distributed load of 1500 lb. per ft. Find the maximum shearing unit-stress. Where in the beam does this stress occur?

Ans. $s_s = 328$ lb. per sq. in.

118. A Douglas fir beam (Fig. 117) having a cross-section 8 in. by 16 in. when tested as a simple beam with a span of 15 ft. by equal concentrated loads at the third points, failed by shear (as shown in Fig. 117) when the total load on the beam was 48,850 lb. Find the maximum shearing unit-stress in the beam when failure occurred. *Ans.* $s_s = 288$ lb. per sq. in.

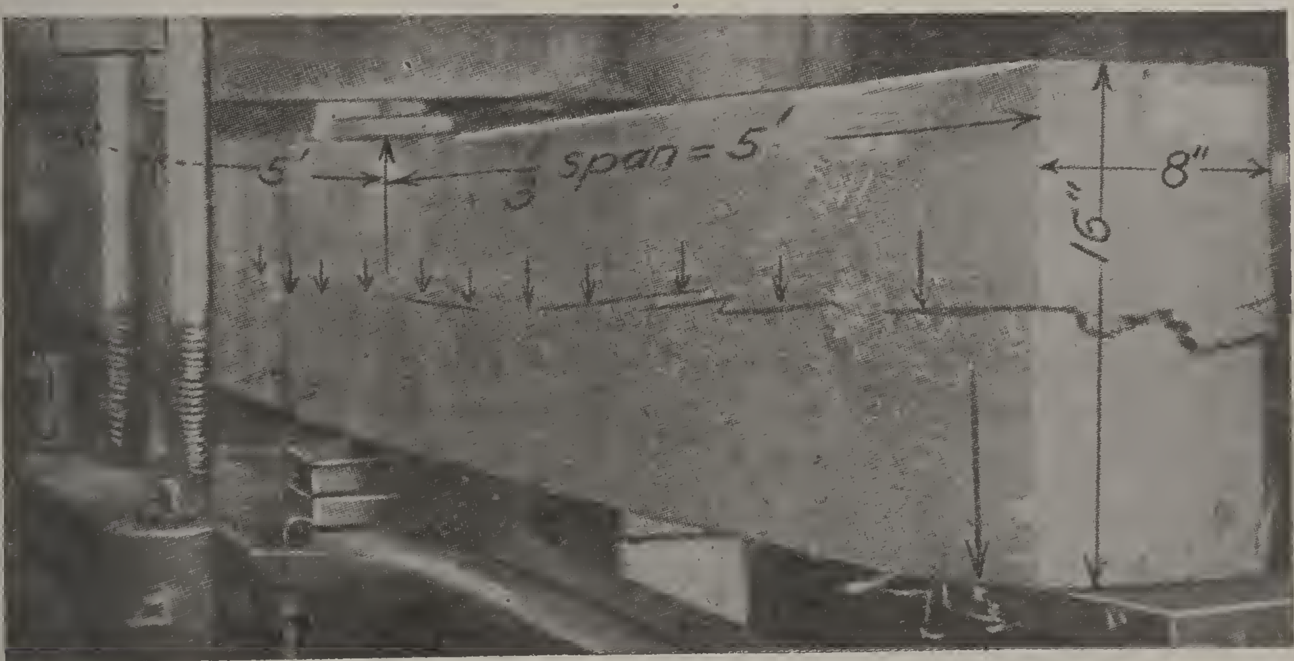


FIG. 117.—Timber beam that failed by horizontal shear.

119. Calculate the maximum shearing unit-stress in a 15-in. 60-lb. I-beam having a span of 12 ft. when loaded with a uniformly distributed load which causes a maximum tensile unit-stress of 16,000 lb. per sq. in.

Ans. $s_s = 4800$ lb. per sq. in.

120. A large Douglas-fir beam is to be designed to resist a concentrated load of 20,000 lb. at the center of the span. Use a working stress equal to one-third of the ultimate shearing unit-stress found in Problem 118 (288 lb. per sq. in.). If the span of the beam is 12 ft. and the width of the beam is to be 8 in. what should be the depth of the beam? A working shearing stress of 100 lb. per sq. in. is frequently specified for large Douglas-fir beams, and 125 lb. per sq. in. for dense pine.

41. Stress Beyond Proportional Limit. Modulus of Rupture.—As shown in Art. 34 the value of s in the flexure formula, $M=\frac{sI}{c}$, is the tensile or compressive unit-stress in the beam *only when s does not exceed the proportional limit of the material.* If the bending moment M causes a unit-stress greater than the proportional limit of the material the bending moment is still held in equilibrium

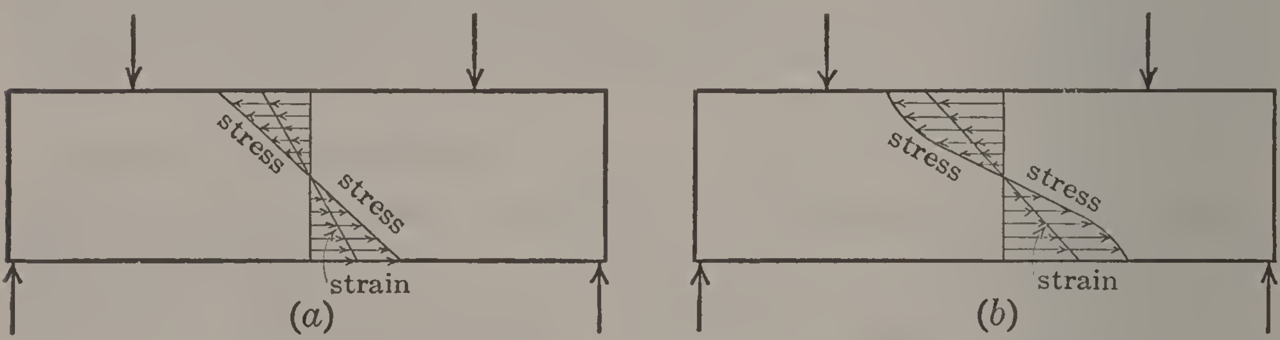


Fig. 118.—Change in distribution of stress when proportional limit is exceeded.

by the resisting moment but the resisting moment is not given by the expression $\frac{sI}{c}$ since this expression was found by assuming that the unit-stress at any point varies directly as the distance of the point from the neutral axis; and this is true only when the maximum unit-stress does not exceed the proportional limit.

The distribution of stress on the cross-section of a beam when the maximum unit-stress on the section is less and greater than the proportional limit are shown in Figs. 118(a) and (b), respectively, and may be explained as follows: Experiments have shown that a plane section of a beam remains approximately plane after the beam is stressed even though the unit-stress exceeds the proportional limit of the material, and hence the unit-strain at any point is proportional to the distance of the point from the neutral axis in both beams of Fig. 118. In the beam that is stressed beyond the proportional limit (Fig. 118b) the unit-stress is not proportional to the unit-strain except near the neutral axis where the strain is small, and hence the unit-stress does not vary directly as the distance from the neutral axis except for short distances from the neutral axis; when the unit-stress on any fiber exceeds the proportional limit the unit-stress on that fiber increases at a less rate than does the strain; the curve showing the distribution of unit-stress on the section is the same as a portion of the stress-stress diagram for the material.

Modulus of Rupture.—From Fig. 118 it is evident that with a given unit-stress in the outer fiber of the beam the resisting moment is larger when the unit-stress varies as shown in Fig. 118(*b*) than it would be if the unit-stress varied directly as the distance from the neutral axis. Therefore, the resisting moment in a beam when stressed to a given unit-stress beyond the proportional limit is larger than that found from the expression $\frac{sI}{c}$ by substituting the given unit-stress for s. Thus, if the ultimate strength of the material is substituted for s in the flexure formula $M=\frac{sI}{c}$, the resulting value of M will be less than the maximum moment that the beam can resist.

Tests of beams of various material have shown that the maximum bending moment a beam can resist may be from 20 to 100 per cent greater, depending on the material and the shape of cross-sections, than the value of M found from the flexure formula by substituting the ultimate strength of the material for s.

If, then, the allowable or working unit-stress in a beam is taken to be some proportion ($\frac{1}{5}$, say) of the tensile or compressive ultimate strength (using the lesser value) the maximum bending moment that could be applied to the beam would be greater than five times the moment that would produce the working stress. For some materials (particularly brittle materials) a working unit-stress based on the ultimate strength of the material seems unnecessarily small and this fact is sometimes employed to justify larger working stresses in beams than is used for tension members. The value of s found from the flexure formula by substituting for M the value of the maximum bending moment that a beam resists when tested to rupture is called the *modulus of rupture* of the material in flexure. Thus, if it is denoted by s_r, we have

$$s_r=\frac{M_{\max.}c}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

It should be noted that s_r is not the unit-stress in the material caused by the bending moment $M_{\max}$; it is not the ultimate strength of the material.

But although the modulus of rupture is not the actual maximum unit-stress in the beam the relative values of the actual unit-stresses in two beams when stressed beyond the proportional limit may be found from the moduli of rupture provided that the two beams have cross-sections of the same shape; and only slight error will occur if the beams have cross-sections of different shapes provided that the cross-sections are symmetrical with respect to the neutral axis and that the stress-strain curves of the material in tension and compression have the same slopes (moduli of elasticity in tension and compression are equal). If these two conditions are not satisfied the neutral axis shifts when the beam is stressed beyond the elastic limit and does not pass through the centroidal axis of the section.

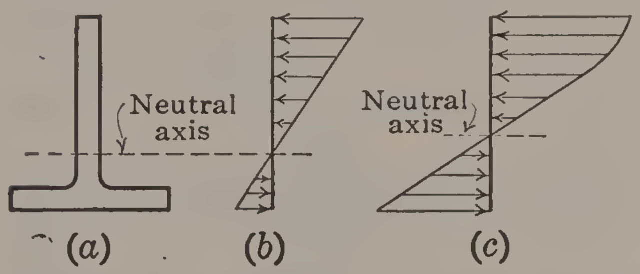


Fig. 119.—Shifting of neutral axis in unsymmetrical section when proportional limit is exceeded.

For example, in a beam with an unsymmetrical section (Fig. 119*a*), the most remote fibers on one side of the neutral axis will be stressed to their proportional limit before those on the other side, as suggested by Fig. 119(*b*), and after the proportional limit has been exceeded, in order to keep the sum of the stresses on the fibers above the neutral axis equal to the sum of the stresses on the fibers below the neutral axis, the neutral axis must shift somewhat, as indicated in Fig. 119(*c*), thereby changing the values of c and I.

Further, if the beam has a symmetrical section but is made of material, such as cast iron, whose fibers deform more in tension than in compression when subjected to a given unit-stress, the neutral axis must shift toward the compressive side of the beam in order to keep the sum of the tensile stresses equal to the sum of the compressive stressses, and this shifting of the neutral axis increases the resistance of the beam.

42. Maximum Moment Due to Moving Loads.—Let several concentrated loads move or roll over a beam so that the distance between the loads remain constant, as for example, the wheel loads of a locomotive when moving on a bridge, and let it be required to find the position of the loads when they cause the greatest bending moment on the beam.

When the loads are in any given position on the beam the dangerous section must occur under one of the loads since the vertical shear will pass through zero under one of the loads, but although the bending moment is maximum at a section beneath a certain one of the loads (W_2, say, Fig. 120) when the loads are in a given position, the bending moment at a section beneath the same load (W_2) after the loads have been moved to another position may be greater or less than it was when the loads were in the previous position.

In Fig. 120 let W_1, W_2, W_3, W_4 and W_5 be concentrated loads that remain fixed distances apart as they roll over the beam of span l, and let it be required to find the position of W_2 such that the bending moment at a section beneath W_2 will be greater than for any other position of W_2.

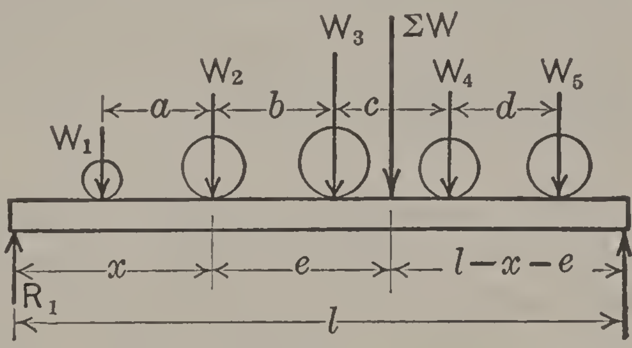


FIG. 120.—Moving loads.

The resultant of the loads is ΣW and it will be assumed to act at the distance e from W_2. Now from one of the equations of equilibrium we have,

$$R_1=\frac{l-x-e}{l}\Sigma W.$$

Let x denote the distance of W_2 from the left support, then the bending moment at the section beneath W_2 is

$$M_x=\frac{l-x-e}{l}x\Sigma W-W_1a.$$

Now the value of x which will make M_x a maximum may be found by equating to zero the first derivative of M_x with respect to x. Thus,

$$\frac{dM_x}{dx}=\frac{l-2x-e}{l}\Sigma W=0.$$

Hence

$$\frac{l-2x-e}{l}=0,$$

or

$$x=\frac{l}{2}-\frac{e}{2}, \quad . \quad . \quad . \quad . \quad (56)$$

and hence M_2 *will be in its position of maximum moment when it is as far to one side of the center of the span as the resultant of all the loads on the span is to the other side of the center of the span.* And the same statement will apply to each of the loads.

In order to find the greatest bending moment to which the beam is subjected each load in turn must be put in its position of maximum bending moment and the bending moment at the section beneath the load must be found; the greatest of these maximum moments is the *greatest* bending moment to which the beam is subjected.

PROBLEM

121. Two loads, 4000 lb. and 2000 lb., 6 ft. apart roll over a simple beam 12 ft. long. Find the position of the loads to give the maximum bending moment and design a yellow-pine beam to carry this load, using a working stress of 1200 lb. per sq. in. *Ans.* 8 in. by 10 in.

43. Assumptions and Limitations Involved in the Flexure Formula.—In the derivation of the flexure formula, $M=\frac{sI}{c}$, several assumptions were made which impose limitations on the use of the formula. These assumptions and limitations may be summarized as follows:

1. The unit-stress on any fiber of the beam is proportional to the unit-strain of the fiber, and hence the maximum fiber unit-stress in the beam does not exceed the proportional limit. This assumption of proportionality of stress and strain is a close approximation to the law of behavior of most engineering materials except cast iron and concrete. And, for these materials the assumption does not as a rule introduce serious error in the flexure formula when the unit-stress in the beam is not greater than the usual working stress.

2. The beam is composed of material for which the modulus of elasticity in tension is the same as that in compression. This assumption is a reasonably close approximation to the results of experiments for most engineering materials, except cast iron. And the error involved in the case of cast iron, although considerable, is not as a rule serious. The flexure formula does not, of course, apply directly to a beam made of two or more different materials such as a reinforced concrete beam.

3. The axial strain (stretch or shortening) of any fiber in the

beam is proportional to the distance of the fiber from the neutral surface. This involves the further assumption that the effect of shearing strain on the axial strain is negligible; which, except for very short deep beams, introduces little error, particularly since the maximum longitudinal fiber stress occurs at the section on which the shearing stress is zero.

4. The *unit*-strain as well as the total strain of any fiber of the beam is proportional to the distance of the fiber from the neutral axis. This assumption requires that all fibers shall have the same length before bending, that is, the beam shall be straight, and hence the flexure formula does not apply to curved beams.

5. The loads act in one plane which contains the centroidal axis of the beam; the loads are perpendicular to the centroidal axis of the beam; and the neutral surface is perpendicular to the plane of the loads. These assumptions require that the plane of the loads shall contain an axis of symmetry of each cross-section (or, to be more exact, the plane of the loads shall contain a principal axis of inertia of each cross-section) in which case the neutral axis of any section is the other axis of symmetry (or the other principal axis of inertia of the cross-section). The flexure formula does not apply, therefore, to a beam loaded unsymmetrically.

6. The proportions of the beam are such that the beam acts as a unit, that is, the beam fails (elastically) by bending and not by twisting, lateral collapse, local wrinkling, etc. For example, a rectangular beam $\frac{1}{4}$ in. wide by 12 in. deep would probably fail by twisting, and an I-beam having very wide and thin flanges would probably fail by local wrinkling of the flange, etc.

Most of the steel, timber and other one-material beams that occur in structures and machines conform approximately to the conditions on which the flexure formula is based. Further, the flexure formula is applied frequently to beams (such as curved beams, unsymmetrically loaded beams, flat plates, etc.), that do not satisfy the conditions required to make its application valid; this practice is often justified if the amount of the error is known approximately, and if allowance is made for the error in some way as, for example, by reducing the working stress, or by introducing a correction factor in the formula.

CHAPTER VI

DEFLECTION OF STATICALLY DETERMINATE BEAMS

(Double Integration Method [1])

44. Introduction.—Beams in structures and machines when resisting static loads must have stiffness as well as strength; that is, beams must resist the loads without permitting too great an elastic deflection.[2] In fact stiffness is sometimes the governing factor in the design of the beam.

Now the deflection of a beam depends on (1) the loads acting on the beam, (2) the stiffness of the material of which the beam is made, and (3) the dimensions of the beam. The stiffness of a material is measured by the modulus of elasticity of the material (Arts. 5 and 144).

The purpose of this chapter is to determine in what way and to what extent the elastic deflection of a beam with two supports (statically determinate beam) depends on the loads, the stiffness of the material, and the dimensions of the beam, when the beam is loaded in various ways. To do this the general equation of the elastic curve of a beam will first be found.

45. Elastic Curve Equation.—The elastic curve of a beam is the curve of the centroidal axis of a stressed beam, provided that the maximum unit-stress in the beam does not exceed the proportional limit of the material; since the elastic curve lies in the neutral surface of the beam it does not change its length as the beam is bent.

The general equation of the elastic curve of a beam may be derived as follows: Fig. 121 represents a beam [3] bent by loads that

[1] See Chapter VIII for a discussion of the moment-area method of determining deflections of beams.

[2] As discussed in Chapter XIII, the resistance of a member to impact loads may be decreased by making the member stiff.

[3] It is assumed that the beam is straight before the loads are applied and that the plane of the loads contains an axis of symmetry of each cross-section

cause a maximum fiber stress less than the proportional limit of the material (the deflection of the beam is exaggerated in the figure). As discussed in Art. 34, when a straight beam is bent the strains of the fibers are proportional to the distances of the fibers from the neutral surface. Thus in Fig. 121 GH and $E'F'$ are any two sections that were parallel before the beam was bent; the distance $\overline{AB}$ between the two parallel sections is assumed to be a differential length and will be denoted by dl, where l is the length of

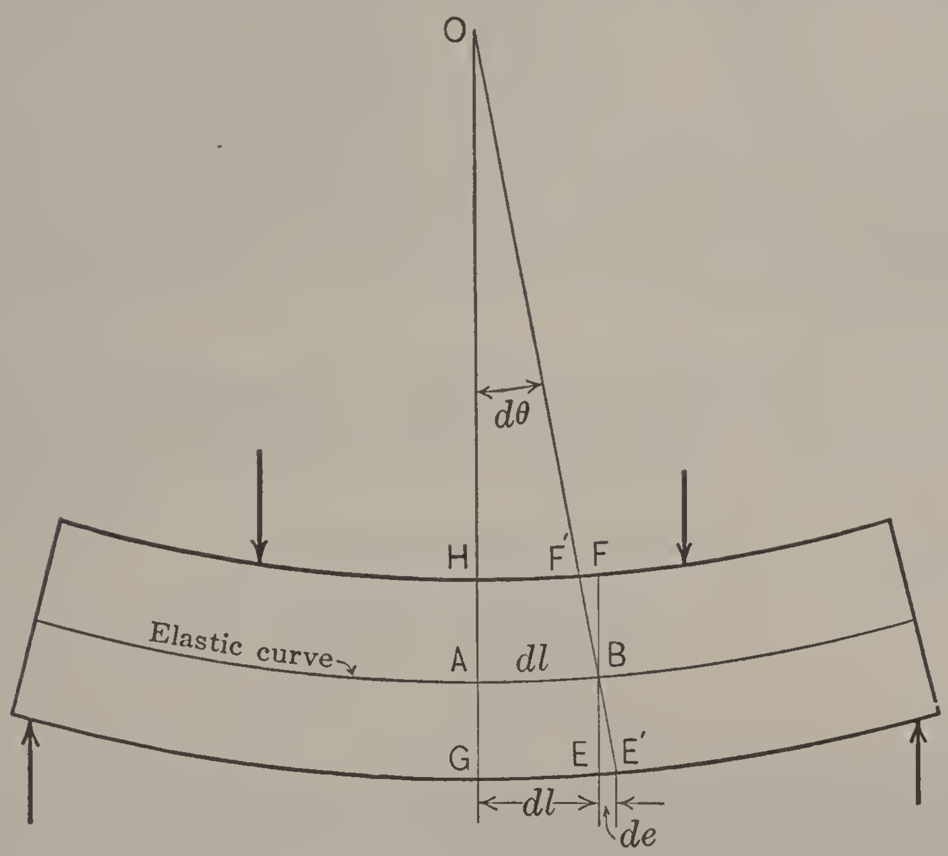


FIG. 121.—Deflection of a beam.

the elastic curve. By drawing EF through B parallel to GH (and hence parallel to the original position of $E'F'$) it is evident that the upper fiber $\overline{HF}$ has shortened the amount $\overline{FF'}$ and the bottom fiber has elongated the amount $\overline{EE'}$, denoted by de in Fig. 121, and the strains of the other fibers are proportional to the distances of the fibers from the neutral surface. Further, the unit-strain, ϵ, of the bottom fiber is

$$\epsilon = \frac{de}{dl}.$$

of the beam so that the neutral plane is perpendicular to the plane of loads (see Art. 43). Further, the beam is assumed to be subjected to pure bending; that is, the deflection resulting from shearing deformation is assumed to be negligible (see Art. 43).

Now, the sectors OAB and BEE' are similar, and hence

$$\frac{\overline{BE}}{\overline{OA}}=\frac{\overline{EE}}{\overline{AB}}$$

But $\overline{OA}$ is the radius of curvature of the elastic curve at the point A, denoted by ρ, and $\overline{BE}$ is the distance from the neutral surface to the fiber whose strain is de, denoted by c as in Art. 34. Hence

$$\frac{c}{\rho}=\frac{de}{dl}=\epsilon; \quad \rho=\frac{c}{\epsilon}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

But c and ϵ are related to the external forces, the stiffness of the material, and the dimensions of the beam according to the following equations:

$$M=\frac{sI}{c} \text{ (Art. 34)} \quad \text{and} \quad E=\frac{s}{\epsilon} \text{ (Art. 5).}$$

Therefore, equation (57) may be written:

$$\rho=\frac{EI}{M} \quad \text{or} \quad M=\frac{EI}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

in which ρ is the radius of curvature of the elastic curve at a section for which the bending moment is M; E is the modulus of elasticity of the material, and I is the moment of inertia of the cross-section of the beam about the neutral axis; if E is expressed in pounds per square inch, I in inches^4 and ρ in inches, M will be expressed in pound-inches.

It should be noted from the above equation that if a beam of constant cross-section is so loaded that the bending moment M is constant over a portion of the beam, the radius of curvature of the elastic curve of this portion will also be constant (since E and I are constant), and hence the elastic curve for this portion is an arc of a circle. Conversely, if a beam is bent in an arc of a circle the bending moments for all sections of the beam are equal. Further, the above equation also shows that when M is equal to zero, ρ is equal to infinity; thus, at the inflection point ($M=0$, Art. 38) the center of curvature is at an infinite distance from the beam.

PROBLEMS

122. A band saw 1/20 in. thick by 1 in. wide is in contact with the circumference of a wheel having a diameter of 50 in. Find the bending moment to which the saw is subjected and the unit-stress in the outer fibers.

Ans. $s=30{,}000$ lb. per sq. in.

123. A steel rod is subjected to bending couples at its ends as shown in Fig. 122. If the moment of each couple is 2400 lb.ft. and the cross-section of the rod is $\frac{1}{2}$ in. by 2 in., with the 2-in. dimension in the plane of the couple, find the radius of curvature of the rod.

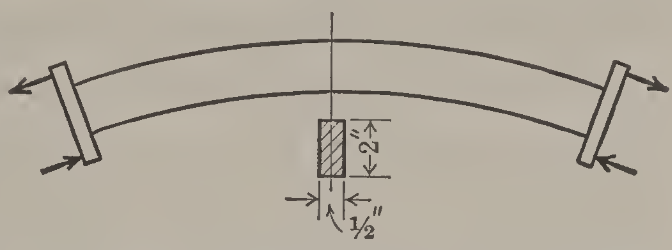


FIG. 122.—Beam subjected to bending couples.

124. Find the radius of curvature at the section AB of the beam described in Problem 73, assuming the beam to be made of yellow pine.

Elastic Curve Equation Expressed in Rectangular Coordinates. The expression for the radius of curvature, ρ, is (see any textbook on calculus)

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}. \qquad (59)$$

Now for beams that are straight before being bent and that are not deflected more than is usual in structural and machine members, the value of the slope $\frac{dy}{dx}$ is always small [4] compared with unity, and hence $\left(\frac{dy}{dx}\right)^2$ may be neglected without introducing serious error. Thus, the above expression for ρ becomes

$$\rho=\frac{1}{\frac{d^2y}{dx^2}}. \qquad (60)$$

Therefore, the equation $M=\frac{EI}{\rho}$ may be written

$$M=\pm EI\frac{d^2y}{dx^2}, \qquad (61)$$

[4] For example, the tangent to the elastic curve would probably never be as large as one part in 20 and hence the error in equation (60) would not exceed about one part in 10,000.

which is the general equation of the elastic curve of a beam. In this equation M is the bending moment at the section whose distance from the origin of the coordinates is x, and y is the deflection of the elastic curve at the same section. The sign to be selected for the right-hand member of the equation is discussed below.

In order to determine the value of the deflection y for any given value of x, M is expressed in terms of x and the differential equation is then integrated twice; the equation thus found, of course, will depend on the type of beam (simple, cantilever, etc.) and the type of loading (concentrated, uniformly distributed, etc.). The equations of the elastic curves and the maximum deflections of beams of various types are found in Art. 46 to 50.

Signs of M and $\frac{d^2y}{dx^2}$.—In using the elastic curve equation, $M = \pm EI\frac{d^2y}{dx^2}$, in this and the following chapters, it is important to understand the significance of the signs of M and $\frac{d^2y}{dx^2}$; E and I are essentially positive and may be regarded merely as magnitudes. Now the sign of M has already been discussed (Art. 36); it is positive for a horizontal beam when it produces tensile stress in the bottom fibers of the beam, or when it causes the center of curvature to lie above the beam, and is negative when it causes compressive stress on the bottom fibers, etc. The sign of $\frac{d^2y}{dx^2}$, however, depends on the choice of the positive direction of the

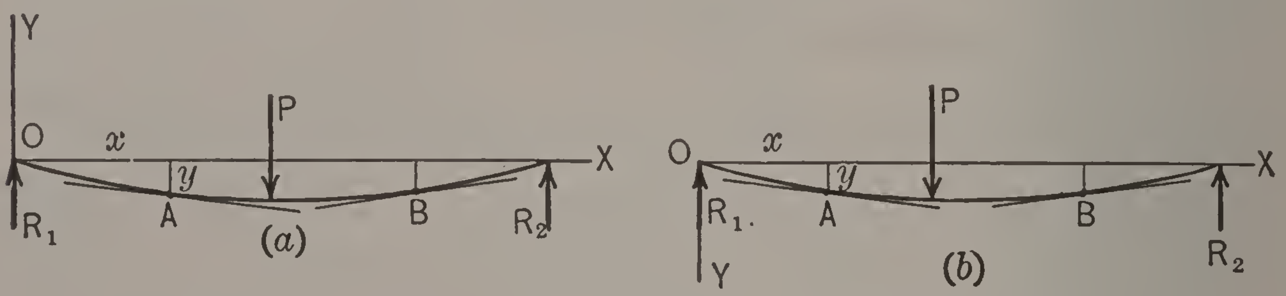


FIG. 123.—Effect of direction of axes on sign of $\frac{d^2y}{dx^2}$.

axes. For example, Fig. 123*a* represents a horizontal simple beam subjected to a positive bending moment. Let the origin of axes be chosen at the left end of the beam and let the positive directions for the x and y axes be to the right and upwards, respectively.

Now the slope $\frac{dy}{dx}$ at a point A on the curve is negative whereas at a point B the slope is positive, and thus as x increases $\frac{dy}{dx}$ increases. Therefore, the rate of change of $\frac{dy}{dx}$ with respect to x $\left(\text{that is, } \frac{d^2y}{dx^2}\right)$ is positive, and since M is also positive, the equation may be written

$$M = EI\frac{d^2y}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

If, however, the positive direction of the y-axis were chosen downward as shown in Fig. 123(b), the slope at A would be positive and would decrease as x increases; thus $\frac{d^2y}{dx^2}$ would be negative. But M is positive and the right side of the equation must, then, also be posi i e which requires that the equation shall be written

$$M = -EI\frac{d^2y}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

If, then, the x-axis is chosen as in Fig. 123 and the positive direction of the y-axis is chosen opposite to the direction of the deflection, equation (62) should be used since the sign of $\frac{d^2y}{dx^2}$ will be the same as that of M, and if the positive direction of the y-axis is chosen opposite to the direction of the deflection equation (63) should be used, since the sign of $\frac{d^2y}{dx^2}$ will be opposite to that of M.

46. Deflection of Simple Beam, Uniform Load.—Let it be required to determine the equation of the elastic curve and the maximum deflection of a simple beam, having a span of l ft., when subjected to a uniformly distributed load of w lb. per ft. (Fig. 124); the known quantities in addition to w and l are the modulus of elasticity E and the moment of interia, I, of the cross-section of the beam.

Let the y-axis be chosen positive upward as shown in Fig. 124. The general elastic curve equation, then, is $EI\frac{d^2y}{dx^2} = M$. But M varies with x and may be expressed in terms of x; thus, the bending, moment at any section whose distance from the left support is x, is

$$M_x = R_1 x - wx \cdot \frac{x}{2} = \frac{wl}{2}x - \frac{wx^2}{2}.$$

Substituting this expression for M in the general elastic curve equation we have

$$EI\frac{d^2y}{dx^2} = \frac{wl}{2}x - \frac{wx^2}{2}. \quad \ldots \ldots \quad (64)$$

By integrating we obtain,

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + c_1,$$

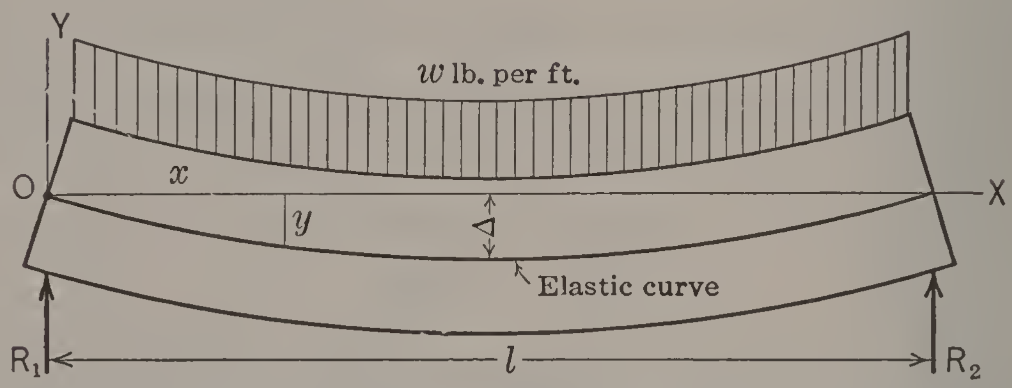


Fig. 124.—Deflection of simple beam subjected to uniform load.

in which c_1 is a constant of integration. The value of a constant in an equation may be determined by substituting a pair of values of the variables; in this equation the variables are the slope $\left(\frac{dy}{dx}\right)$ of the elastic curve and the distance x. From inspection,

when $$x = \frac{l}{2}, \quad \frac{dy}{dx} = 0.$$

By substituting this pair of values in the above equation the value of c_1 is found. Thus

$$c_1 = -\frac{wl^3}{16} + \frac{wl^3}{48} = -\frac{1}{24}wl^3.$$

The above equation, then, becomes,

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{1}{24}wl^3. \quad \ldots \ldots \quad (65)$$

By integrating this equation and evaluating the constant of integration, the equation expressing the relation between y and x is found. Thus,

$$EIy=\frac{wlx^3}{12}-\frac{wx^4}{24}-\frac{wl^3x}{24}+c_2.$$

From inspection, when $x=0$, $y=0$, and hence $c_2=0$.

Therefore, the equation of the elastic curve of a simple beam subjected to a uniformly distributed load is

$$EIy=\frac{wlx^3}{12}-\frac{wx^4}{24}-\frac{wl^3x}{24}. \quad (66)$$

Maximum Deflection.—Now the maximum deflection, Δ, occurs at the mid-span, that is, y in the above equation becomes Δ when x is equal to $l/2$. Hence the maximum deflection is

$$\Delta=\frac{1}{EI}\left(\frac{wl^4}{96}-\frac{wl^4}{384}-\frac{wl^4}{48}\right).$$

$$\Delta=-\frac{5}{384}\frac{wl^4}{EI}=-\frac{5}{384}\frac{Wl^3}{EI}. \quad (67)$$

The minus sign shows that the deflection is opposite to the positive direction of the y-axis. If the total load W is expressed in pounds, l in inches, E in pounds per square inch, and I in inches^4, Δ will be expressed in inches.

47. Deflection of Simple Beam, Concentrated Load at Mid-span.—Let the axes be chosen as shown in Fig. 125. For any section in the left half of the beam,

$$M=R_1x=\tfrac{1}{2}Px.$$

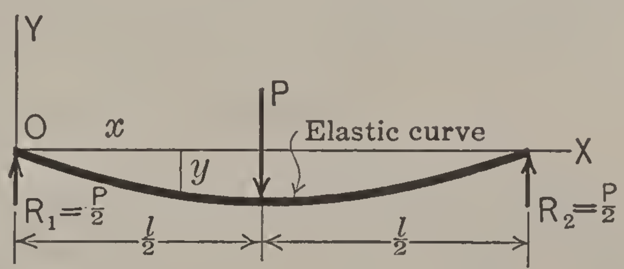


FIG. 125.—Deflection of simple beam subjected to concentrated load at mid-span.

Hence

$$EI\frac{d^2y}{dx^2}=\tfrac{1}{2}Px. \quad (68)$$

By integrating we obtain

$$EI\frac{dy}{dx}=\frac{Px^2}{4}+c_1;$$

when $x=\frac{l}{2}$, $\frac{dy}{dx}=0$, and hence $c_1=-\frac{Pl^2}{16}$.

Thus,

$$EI\frac{dy}{dx}=\frac{Px^2}{4}-\frac{Pl^2}{16}. \qquad (69)$$

By integrating again we obtain

$$EIy=\frac{Px^3}{12}-\frac{Pl^2}{16}x+c_2; \qquad (70)$$

when $x=0$, $y=0$, and hence [5] $c_2=0$.

Therefore, the equation of the elastic curve of the left half of the beam is

$$EIy=\frac{Px^3}{12}-\frac{Pl^2x}{16}. \qquad (71)$$

Since the maximum deflection, Δ, occurs at the mid-span, that is, when $x=\frac{l}{2}$, the maximum deflection is

$$\Delta=\frac{Pl^3}{96}-\frac{Pl^3}{16}=-\frac{1}{48}\frac{Pl^3}{EI}.$$

The minus sign shows that the deflection is opposite to the positive direction of the Y-axis.

Equation (71) is also the equation of the elastic curve of the right half of the beam if the origin of axes is taken at the right end of the beam and the positive direction of x is to the left.

PROBLEMS FOR ARTICLES 46 AND 47

125. A simple beam 12 ft. long is subjected to a uniformly distributed load of 400 lb. per ft. The cross-section of the beam is 4 in. wide by 8 in. deep. The beam is made of oak having a modulus of elasticity of 2,000,000 lb. per sq. in. and a proportional limit of 3500 lb. per sq. in. It is specified that the maximum fiber unit-stress shall not exceed 2000 lb. per sq. in. and that the maximum deflection shall not exceed $\frac{1}{360}$ of the span. Are the requirements satisfied?

126. How much will a 12-in. 55-lb. I-beam deflect when stressed to its porportional limit of 40,000 lb. per sq. in. by a concentrated load at the mid-span if the span is 20 ft.? *Ans.* $\Delta \doteq 1.07$ in.

[5] It should be noted that although the right support does not deflect, the value of y in equation (70) is not zero when x is equal to l, since $l/2$ is the greatest value x can have in the equation; for values of x greater than $l/2$ the expression for M is not $\frac{1}{2}Px$.

127. A cylindrical steel shaft 4 in. in diameter is used as a simple beam on a span of 12 ft. The beam is subjected to a uniformly distributed load of 200 lb. per ft. and a concentrated load of P lb. at the center of the span. If the maximum deflection of the beam must not exceed 0.40 in. and the maximum fiber unit-stress must not exceed 20,000 lb. per sq. in., find the maximum value of P. *Ans.* $P=2290$ lb.

128. In Art. 47 use for M the bending moment for any section to the *right* of the load and show that the magnitude of the maximum deflection is $\frac{1}{48}\frac{Pl^2}{EI}$.

48. Deflection of Cantilever Beam, Uniform Load.—Let the uniformly distributed load be w lb. per ft. and let the axes be chosen as in Fig. 126.

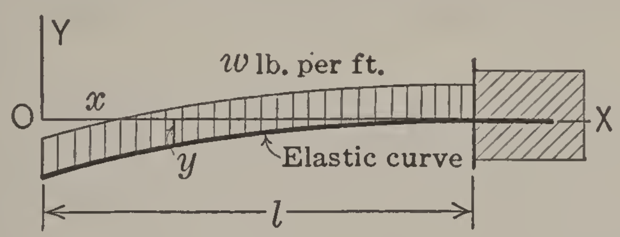


FIG. 126.—Deflection of cantilever beam subjected to uniform load.

Since M is negative the elastic curve of the beam is

$$EI\frac{d^2y}{dx^2}=-M=-\frac{wx^2}{2}. \qquad (72)$$

By integrating and determining constants as in the preceding articles we have

$$EI\frac{dy}{dx}=-\frac{wx^3}{6}+c_1,$$

when $x=l$, $\frac{dy}{dx}=0$; therefore, $c_1=\frac{wl^3}{6}$.

$$EI\frac{dy}{dx}=-\frac{wx^3}{6}+\frac{wl^3}{6}. \qquad (73)$$

$$EIy=-\frac{wx^4}{24}+\frac{wl^3}{6}x+c_2,$$

when $x=l$, $y=0$; therefore, $c_2=-\frac{1}{8}wl^4$.

The elastic curve equation then is

$$EIy=-\frac{wx^4}{24}+\frac{wl^3}{6}x-\frac{1}{8}wl^4. \qquad (74)$$

Now the maximum value of y in the above equation occurs when x is equal to 0. Hence the maximum deflection is

$$\Delta = -\frac{1}{8}\frac{wl^4}{EI} = -\frac{1}{8}\frac{wl^3}{EI}. \quad \ldots \ldots \quad (75)$$

PROBLEMS

129. Show that the magnitude of the maximum deflection of a cantilever beam when subjected to a concentrated load P at its free end is

$$\Delta = \frac{1}{3}\frac{Pl^3}{EI}.$$

130. A 15-in. 55-lb. I-beam is used as a cantilever beam. The length of the cantilever is 10 ft. and the beam is subjected to a uniformly distributed load. Find the deflection of the free end of the beam when the maximum fiber unit-stress in the beam is 16,000 lb. per sq. in.; neglect the weight of the beam. *Ans.* $\Delta = 0.256$ in.

131. A piece of pine having a cross-section 2 in. wide by 4 in. deep is used as a cantilever beam to resist a concentrated load P at the free end. The beam has a free length of 4 ft. It is required that the maximum deflection (see Problem 129) shall not exceed 0.5 in. and that the maximum fiber unit-stress shall not exceed 2000 lb. per sq. in. What is the maximum value of P? Neglect the weight of the beam. The modulus of elasticity of the pine may be assumed to be 1,500,000 lb. per sq. in.

49. Deflection of Overhanging Beam, Uniform Load.—Fig. 127(a) represents a beam that overhangs the same amount at each end and is subjected t) a uniformly distributed load of w lb. per ft. over the entire length of the beam. Let it be required to find the equation of the elastic curve of the central portion of the beam, and the maximum deflection of the beam.

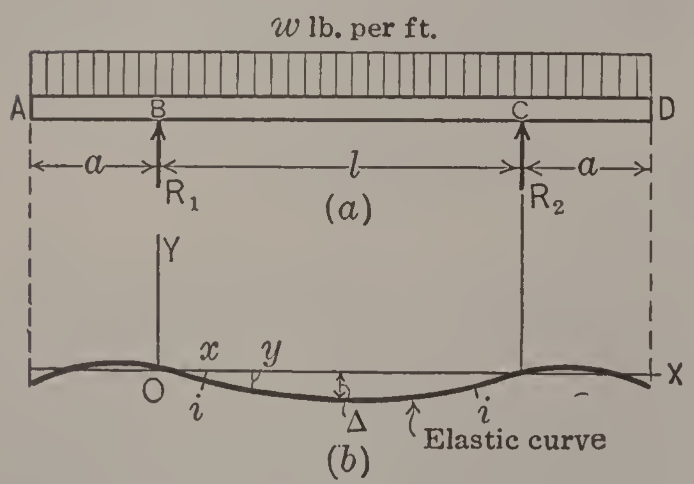


FIG. 127.—Deflection of overhanging beam.

Let the axes be chosen as shown in Fig. 127(b). The reaction R_1 is found by applying one of the equations of equilibrium as follows:

$$\Sigma M_c = 0 = R_1 l + \frac{wa^2}{2} - \frac{w(a+l)^2}{2}.$$

$$R_1 = \tfrac{1}{2}wl + wa.$$

The bending moment at any section between B and C (Fig. 127a) is

$$M_x = R_1 x - \frac{w(a+x)^2}{2}$$
$$= \tfrac{1}{2}w(a+l)x - \tfrac{1}{2}w(a+x)^2. \quad . \quad . \quad . \quad . \quad (76)$$

For all points between B and C:

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}w(l-a)x - \frac{1}{2}wa^2. \quad . \quad . \quad . \quad . \quad (77)$$

By integrating this equation twice and making use of the following conditions,

$$\frac{dy}{dx}=0 \text{ when } x=\frac{l}{2}; \quad y=0 \text{ when } x=0; \quad y=y_{\max}=\Delta \text{ when } x=\frac{l}{2},$$

the elastic curve equation of the central portion of the beam is found to be

$$EIy = -\tfrac{1}{24}wx^4 + \tfrac{1}{12}wlx^3 - \tfrac{1}{4}wa^2x^2 - \tfrac{1}{24}wl^3x + wa^2lx; \quad . \quad (78)$$

and the maximum deflection is found to be

$$\Delta = -\frac{5}{384}\frac{wl^4}{EI} - \frac{3}{16}\frac{wl^2a^2}{EI}. \quad . \quad . \quad . \quad . \quad . \quad (79)$$

50. Deflection of Simple Beam, Concentrated Load Not at Mid-span.—If the concentrated load P is not at the mid-span the expressions for M on opposite sides of the load are different as is also the case when the load acts at the mid-span (see Art. 47), and hence there are two elastic curves which have different equations. But, if the load does not act at the mid-span the constants of integration cannot be determined as in Art. 47; they are found by making use of the fact that the two elastic curves have a common tangent and a common ordinate under the load. Further, the maximum deflection of the beam is the maximum value of y in the equation of only one of the curves.

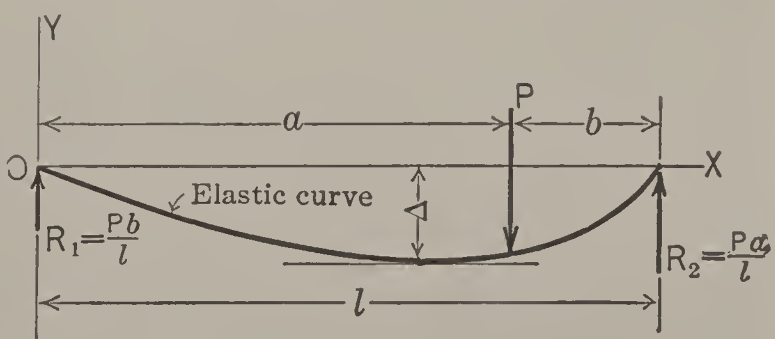


Fig. 128.—Deflection of simple beam; concentrated load at any point.

Fig. 128 represents a simple beam subjected to a concentrated load P at the distance a from the left support and b from the right support, a being greater than b. Let it be required to find the equation of the elastic curve of the left portion of the beam and the maximum deflection of the beam. Let the axes be chosen as shown in Fig. 128.

The bending moment at any section to the left of P is

$$M=R_1x=\frac{Pb}{l}x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

in which x may have any value from 0 to a. And the moment at any section to the right of P is

$$M=\frac{Pb}{l}x-P(x-a), \quad . \quad . \quad . \quad . \quad . \quad (81)$$

in which x can not be less than a nor greater than l.

For all points to the left of the load:

$$EI\frac{d^2y}{dx^2}=\frac{Pbx}{l}. \quad . \quad . \quad . \quad (82)$$

$$EI\frac{dy}{dx}=\frac{Pbx^2}{2l}+c_1. \quad . \quad (84)$$

For all points to the right of the load:

$$EI\frac{d^2y}{dx^2}=\frac{Pb}{l}x-P(x-a). \quad . \quad . \quad (83)$$

$$EI\frac{dy}{dx}=\frac{Pb}{2l}x^2-\frac{P(x-a)^2}{2}+c_3. \quad (85)$$

And since the curves have a common tangent under the load, when x is made equal to a in both (84) and (85) the value of $\frac{dy}{dx}$ in (84) is equal to $\frac{dy}{dx}$ in (85). Thus

$$\frac{Pba^2}{2l}+c_1=\frac{Pba^2}{2l}-\frac{P(a-a)^2}{2}+c_3;$$

whence,

$$c_1=c_3.$$

Now by substituting c_1 for c_3 in (85) and by integrating (84) and (85) there is found:

$$EIy=\frac{Pbx^3}{6l}+c_1x+c_2. \quad (86)$$

when $x=0$, $y=0$,

hence, $c_2=0$.

$$EIy=\frac{Pbx^3}{6l}-\frac{P(x-a)^3}{6}+c_1x+c_4. \quad (87)$$

Since the curves have a common ordinate under the load, when $x=a$ in (86) and (87) the values of y in these equations are equal, and hence

$$\frac{Pba^3}{6l}+c_1a=\frac{Pba^3}{6l}+c_1a+c_4.$$

$$c_4=0.$$

Now when $x=l$ in (87), $y=0$, and hence

$$c_1=-\frac{Pbl^2}{6l}+\frac{P(l-a)^3}{6l}=-\frac{Pb}{6l}(l^2-b^2). \quad . \quad . \quad (88)$$

Substituting this value of c_1 in (86) the equation of the elastic curve of the left portion of the beam is found to be

$$EIy=\frac{Pbx^3}{6l}-\frac{Pb(l^2-b^2)x}{6l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

The value of x that makes $\frac{dy}{dx}$ equal to zero is the value of x that makes y in (89) a maximum, and this value of y is the value of the maximum deflection. But the value of $\frac{dy}{dx}$ is given by (84) and hence equating (84) to zero we have

$$x^2=\frac{l^2-b^2}{3}=\frac{a(a+2b)}{3}.$$

By substituting this value of x in (89) the maximum deflection is found to be

$$\Delta=-\frac{Pb(l^2-b^2)\sqrt{3(l^2-b^2)}}{27EIl}, \quad . \quad . \quad . \quad . \quad (90)$$

$$\Delta=-\frac{Pba(a+2b)\sqrt{3a(a+2b)}}{27EIl}. \quad . \quad . \quad . \quad . \quad (91)$$

If in the above expression a and b are each made equal to $\frac{l}{2}$, that is, if P acts at the mid-span the value of Δ is $\frac{1}{48}\frac{Pl^3}{EI}$ as was found in Art. 47.

PROBLEMS

132. Derive the equations of the elastic curves of a simple beam of length l when loaded with two equal concentrated loads, PP, each load being at a distance of $\frac{1}{4}l$ from a support. Also find the maximum deflection of the beam. Select the origin of axes at the left end of the beam and the positive direction of the x and y axes to the right and upwards, respectively.

$$\textit{Ans.}\ \begin{cases} EIy=\dfrac{Px^3}{6}-\dfrac{3}{32}Pl^2x \text{ for the left portion,} \\[2ex] EIy=\dfrac{Plx^2}{8}-\dfrac{Plx}{8}+\dfrac{Pl^3}{384} \text{ for the central portion,} \\[2ex] \Delta=-\dfrac{11}{384}\dfrac{Pl^3}{EI}. \end{cases}$$

NOTE.—The elastic curves of the left and central portions of the beam have a common tangent and a common ordinate under the left load.

133. A 9-in. 30-lb. I-beam rests on two supports 14 ft. apart and resists a concentrated load of 15,000 lb. at a distance of 5 ft. from the left support. Determine whether the following requirements are satisfied.

(*a*) Fiber unit-stress not to exceed 16,000 lb. per sq. in.

(*b*) Maximum deflect not to exceed $\frac{1}{360}$ of the span.

Deflection of Beam Due to Shear

The deflection of a beam due to the shearing stresses in the beam was assumed, in the preceding discussions, to be negligible. In short deep beams, however, the deflection due to shear may require consideration. An approximate value for the deflection due to the shearing stresses may be found as follows:

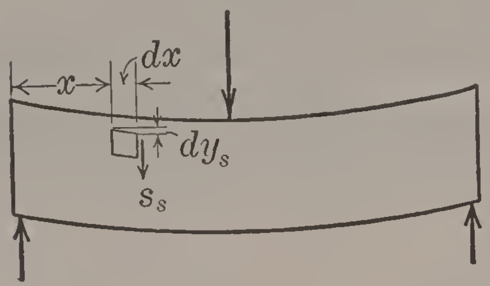


Fig. 129.—Deflection of beam due to shear.

The shearing deformation dy_s of a small block, of length dx, in the beam (Fig. 129) is

$$dy_s = \epsilon_s dx = \frac{s_s}{E_s} dx,$$

in which s_s is the shearing unit-stress on the face of the block; the deflection of a fiber of length x (Fig. 129) is

$$y_s = \frac{1}{E_s}\int_0^x s_s dx. \qquad (92)$$

If s_s is constant throughout the length of all the fibers the deflection of the beam is

$$y_s = \frac{s_s x}{E_s}. \qquad (93)$$

In general, however, s_s varies over the section, but if an average value of s_s (see Art. 40) is used in equation (93), an approximate value of the deflection due to shear may be found which may be helpful in estimating to what extent the shearing stresses contribute to the deflection of a beam.

A more exact analysis may be made by substituting for s_s in equation (92) the value given by equation 53 of Art. 40.

PROBLEM

134. Since $Vdx = dM_x$ (see Arts. 36 and 37) and the average shearing unit-stress $s_s = V/a$ show that equation (93) may be written

$$y_s = \frac{M_x}{aE_s},$$

when the vertical shear, V, is constant over the length x.

CHAPTER VII

STATICALLY INDETERMINATE BEAMS

51. Introduction.—In the preceding two chapters the beams considered were held in quilibrium by external forces that formed a system of parallel forces in a plane, and the reactions of the supports were found by applying the *two* equations of equilibrium for such a force system; since there were not more than two supports and hence not more than two unknown external forces acting on the beam, the *two* equations of equilibrium were sufficient to determine all the unknown external forces. In other words, the force system acting on the simple, cantilever, and overhanging beams considered in the preceding chapters were *statically determinate.*

On the other hand, although fixed-ended beams and continuous beams are held in equilibrium by parallel forces in a plane, yet the reactions of the supports cannot be found from the *two* equations of equilibrium *alone,* since the number of unknown reactions is greater than two. Such beams are said to be *statically indeterminate,* and in finding the reactions use is made of the relation between the external forces and the internal effects of those forces (as expressed by the elastic curve equation) in addition to the relations between the external forces alone (as expressed by the equations of equilibrium).

Method of Procedure.—The method of obtaining equations that involve the unknown reactions, in addition to the two equations of equilibrium, is as follows:

In the elastic curve equation, $M = \pm EI\frac{d^2y}{dx^2}$, the expression for M will contain some of the unknown reactions (forces and couples) acting on the beam, and after the elastic curve equation is integrated, a constant of integration being added with each integration, the unknown reactions and constants of integration are determined by substituting in the equations as many pairs of values of the variables as there are unknown reactions and constants of integration; the pairs of values of the variables are found from the physical conditions that the beam satisfies. The remaining unknown forces (that did not occur in the expression for M) may then, as a rule, be found from the two equations of equilibrium.

Thus, the equation of the elastic curve may be used for two different purposes: (1) To find the deflection of a beam when all the forces acting on the beam are known, as discussed in Chapter VI, and (2) to determine reactions of supports when they cannot be found from the equations of equilibrium alone, as is discussed in this chapter in connection with fixed-ended and continuous beams; the use of the elastic curve equation for the latter purpose is of great importance, since it makes possible the determination of the *stresses* in statically indeterminate flexural members.

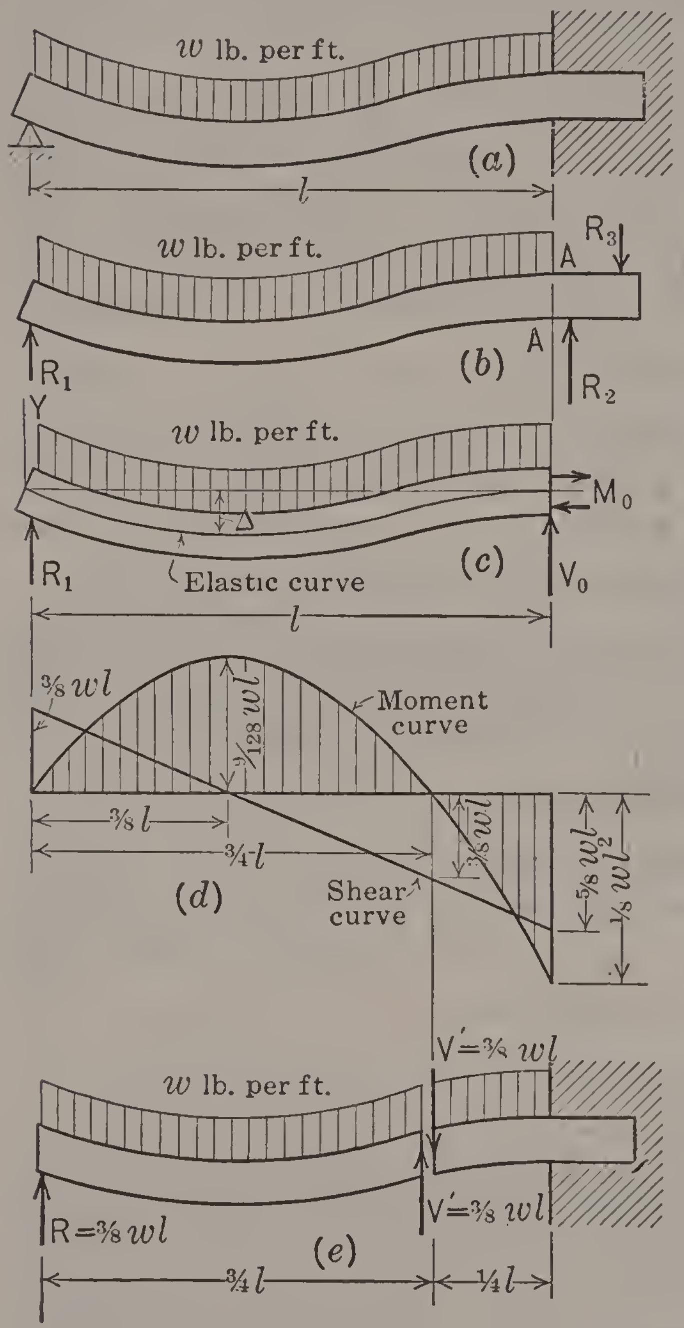


FIG. 130.—Beam fixed at one end, supported at other end; load uniformly distributed.

52. Beam Fixed at One End, Supported at Other End; Uniform Load.—Fig. 130(a) represents such a beam, the tangent line to the elastic curve at the wall being horizontal; the length of span is l, and the load per unit of length is w. As shown in Fig. 130(b), the beam is held in equilibrium by parallel forces of which all three reactions, R_1, R_2 and R_3 are unknown. Thus it is necessary, as explained in the preceding article, to use the elastic curve equation in addition to the two equations of equilibrium in order to determine the reactions.

Now the forces R_2 and R_3 are equivalent to an upward force and a bending couple, and since we are concerned mainly with the part of the beam outside the wall, it is convenient to assume that the beam is cut off even with the face of the wall at the section AA (Fig. 130*b*) and that R_2 and R_3 are replaced by the forces (stresses) which they cause on this section; namely, the upward shearing force V_0 and the bending couple M_0 as shown in Fig. 130(*c*).

The three reactions to be found, then, are R_1, V_0 and M_0. The value of R_1 may be found from the elastic curve equation, and V_0 and M_0 may then be found from the two equations of equilibrium as follows:

If axes are chosen as shown in Fig. 130(*c*) the elastic curve equation (see Art. 45) is

$$EI\frac{d^2y}{dx^2}=M=R_1x-\frac{wx^2}{2}. \quad (94)$$

Integrating, we have

$$EI\frac{dy}{dx}=\frac{R_1x^2}{2}-\frac{wx^3}{6}+c_1,$$

when $x=l$, $\frac{dy}{dx}=0$ and hence,

$$c_1=-\frac{R_1l^2}{2}+\frac{wl^3}{6}.$$

Therefore,

$$EI\frac{dy}{dx}=\frac{R_1x^2}{2}-\frac{wx^3}{6}-\frac{R_1l^2}{2}+\frac{wl^3}{6}. \quad (95)$$

Integrating again, we obtain,

$$EIy=\frac{R_1x^3}{6}-\frac{wx^4}{24}-\frac{R_1l^2}{2}x+\frac{wl^3}{6}x+c^2. \quad (96)$$

when $x=0$, $y=0$ and hence, $c_2=0$;

when $x=l$, $y=0$ and hence,

$$0=\frac{R_1l^3}{6}-\frac{wl^4}{24}-\frac{R_1l^3}{2}+\frac{wl^4}{6}.$$

Therefore,

$$R_1=\tfrac{3}{8}wl.$$

Now from the two equations of equilibrium ($\Sigma F=0$ and $\Sigma M=0$), V_0 and M_0 may be found. Thus,

$$\Sigma F=\tfrac{3}{8}wl+V_0-wl=0.$$

Hence,

$$V_0 = \tfrac{5}{8}wl.$$

$$\Sigma M_B = \tfrac{3}{8}wl \cdot l - wl \cdot \frac{l}{2} + M_0 = 0.$$

Hence,

$$M_0 = -\tfrac{1}{8}wl^2.$$

It will be noted that the moment at the wall, then, is a negative moment, as shown in Fig. 130(c), the magnitude of which is $\frac{1}{8}wl^2$.

Shear and Moment Diagrams. Maximum Positive Moment.—The vertical shear for a section at distance x from the left support is

$$V_x = \tfrac{3}{8}wl - wx,$$

and this equation shows that the shear is zero when $x = \frac{3}{8}l$; the shear diagram is shown in Fig. 130(d).

The bending moment for a section at the distance x from the left support is

$$M_x = \tfrac{3}{8}wlx - wx \cdot \frac{x}{2}. \quad . \quad . \quad . \quad . \quad . \quad (97)$$

The value of the bending moment is a maximum at the section for which the vertical shear is zero (Art. 35), and hence the maximum positive bending moment is

$$\text{Pos. } M_{\max} = \tfrac{3}{8}wl \cdot \tfrac{3}{8}l - w \cdot \tfrac{3}{8}l \cdot \tfrac{1}{2}\,\tfrac{3}{8}l$$
$$= \tfrac{9}{128}wl^2.$$

The inflexion point is found by equating M_x to zero. Thus,

$$\tfrac{3}{8}wlx - \frac{wx^2}{2} = 0,$$

whence,

$$x = \tfrac{3}{4}l.$$

The bending moment diagram is shown in Fig. 130(d).

Since the maximum negative moment $(-\frac{1}{8}wl^2)$, which occurs at the wall, is greater than the maximum positive moment, the greatest unit-stress in this type of beam occurs at the wall and may be found from the flexure formula $\left(M = \frac{sI}{c}\right)$ by using $\frac{1}{8}wl^2$ for M.

Maximum Deflection.—Equation (96) is the equation of the elastic curve, the value of c_2 being zero and the value of R_1 being $\frac{3}{8}wl$. Thus, the elastic curve equation becomes,

$$24EIy = \tfrac{3}{2}wl(x^3 - 3l^2) - w(x^4 - 4l^3x).$$

Further, the maximum deflection is found to occur at the distance $x = 0.4215l$ from the left support and its value is

$$\Delta = 0.0054\frac{wl^4}{EI}.$$

As indicated in Fig. 130(e), the beam may be considered to be a combination of a simple beam having a span $\frac{3}{4}l$ and a cantilever of length $\frac{1}{4}l$. That is, the beam may be sawed in two at the section where the moment is zero (inflection point), the shear V' at this section acting as the reaction for the simple beam and as a concentrated load on the cantilever.

Alternative Method.—If the left support were removed the beam would become a cantilever beam and the uniform load on the beam would cause the free end to deflect a distance $\frac{1}{8}\frac{wl^4}{EI}$ (Art. 48). But since the end does not deflect, the reaction R_1 must be a force which if acting alone on the end of the cantilever would cause an upward deflection equal to $\frac{1}{8}\frac{wl^4}{EI}$. Now the deflection due to a concentrated load P on the free end of a cantilever is $\frac{1}{3}\frac{Pl^3}{EI}$ (Prob. 129). Hence,

$$\frac{1}{3}\frac{R_1l^3}{EI} = \frac{1}{8}\frac{wl^4}{EI}.$$

Therefore,

$$R_1 = \tfrac{3}{8}wl.$$

This method does not, however, avoid the use of the elastic curve equation since the elastic curve equation was used in determining the expressions used for the deflections.

PROBLEMS

134. Derive the expressions for the maximum positive moment, the maximum negative moment, and the maximum deflection, as given above, by considering the beam to be composed of two beams as shown in Fig. 130(e).

135. A steel I-beam is fixed at one end and supported at the other end, the length of span being 10 ft. It is loaded with a uniformly distributed load of 100 lb. per ft. Find (a) the point of inflection, (b) the maximum negative moment, (c) the maximum positive moment, (d) the section modulus of the beam using a working unit-stress of 16,000 lb. per sq. in. Also draw to scale the moment and shear diagrams.

Ans. (d) section modulus = 0.937 in.^{3} if weight of beam is neglected.

136. A timber beam 6 in. wide and 12 in. deep is used on a span of 16 ft. The beam is fixed at one end and supported at the other and is subjected to a uniformly distributed load. If the working unit-stress is 800 lb. per sq. in., what is the maximum load per foot of length the beam can resist.

137. A 10-in. 25-lb. I-beam is supported at the left end and partially restrained in a wall at the right end, the clear span being 15 ft. The load including the weight of the beam is 1200 lb. per ft., and the reaction at the left end is $\frac{5}{12}wl$. Find the fiber unit-stresses in the beam due to the negative and positive bending moments and indicate on a sketch where these stresses occur.

53. Beam Fixed at Both Ends; Uniform Load.—Fig. 131(a) represents such a beam, the length of span being l and the load per unit of length, w. Fig. 131(b) shows that the beam is held in equilibrium by a parallel force system in which there are four reactions, R_1, R_2, R_3 and R_4. But, as in the preceding article, it is desirable to replace these reactions by the shears and moments at the ends of the beams as indicated in Fig. 131(c), and hence the four unknown reactions are V_1, V_2, M_1 and M_2. Now from symmetry[1] and from the equations of equilibrium we have:

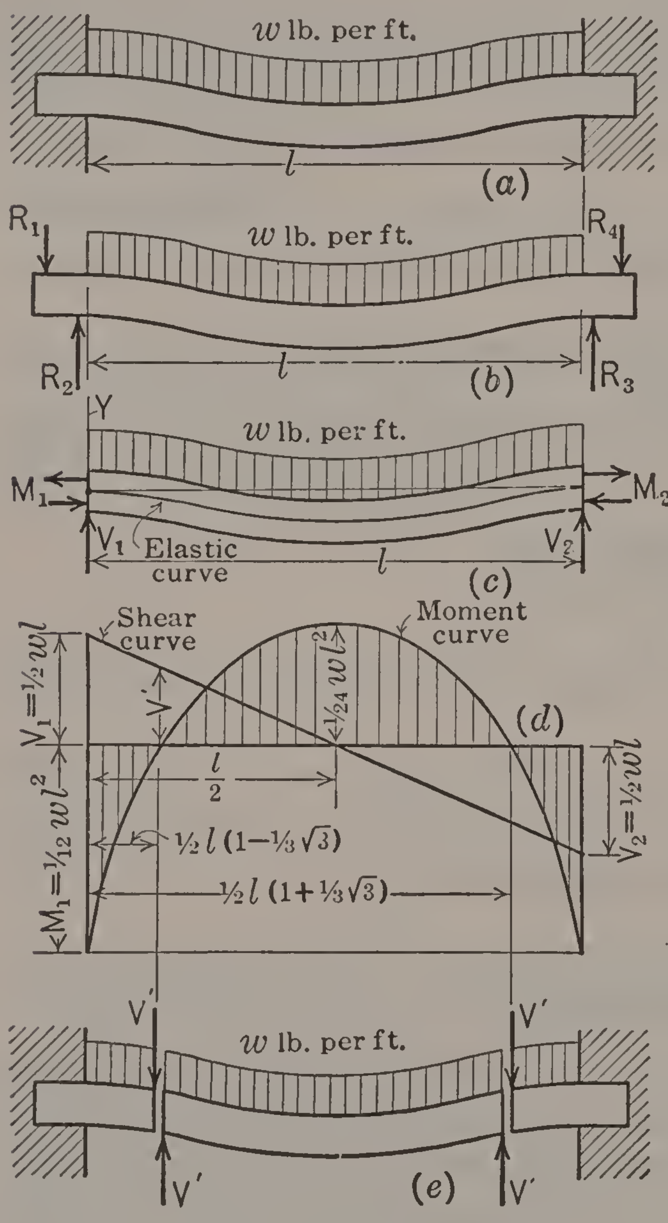


Fig. 131.—Beam fixed at both ends; load uniformly distributed.

$$M_1 = M_2 \text{ and } V_1 = V_2 = \tfrac{1}{2}wl,$$

and hence only one unknown (namely, M_1 or M_2) need be found from the elastic curve equation. The value of M_1 may be found as follows:

[1] If the condition of symmetry were not employed, *two* unknowns (V_1 and M_1) would be found from the elastic curve equation, since only two equations of equilibrium are available.

If axes are chosen as shown in Fig. 131(c) the elastic curve equation is (see Art. 45)

$$EI\frac{d^2y}{dx^2}=M=M_1+V_1x-\frac{wx^2}{2}; \qquad (98)$$

M_1 will be considered to be unknown both in sign and in magnitude. Integrating, we have

$$EI\frac{dy}{dx}=M_1x+\frac{V_1x^2}{2}-\frac{wx^3}{6}+c_1,$$

$$\frac{dy}{dx}=0 \text{ when } x=0, \text{ hence } c_1=0,$$

$$\frac{dy}{dx}=0 \text{ when } x=l, \text{ hence}$$

$$0=M_1l+\tfrac{1}{2}wl\,\frac{l^2}{2}-\frac{wl^3}{6}.$$

Therefore,

$$M_1=-\frac{1}{12}wl^2.$$

Thus the moment at each wall is negative and equal to $\frac{1}{12}wl^2$.

Shear and Moment Diagrams. Maximum Positive Moment.—The vertical shear for a section at the distance x from the left support is

$$V_x=\tfrac{1}{2}wl-wx;$$

this is equal to zero when $x=\frac{1}{2}l$, that is, at the mid-span. The shear diagram is shown in Fig. 131(d).

The bending moment about a section at a distance x from the left support is

$$M_x=M_1+V_1x-\frac{wx^2}{2}$$

$$=-\tfrac{1}{12}wl^2+\tfrac{1}{2}wlx-\frac{wx^2}{2}.$$

But the bending moment is maximum at the section for which the vertical shear is zero and hence,

$$\text{Maximum Positive Moment}=-\tfrac{1}{12}wl^2+\tfrac{1}{2}wl\frac{l}{2}-\frac{wl^2}{8}$$

$$=\frac{1}{24}wl^2.$$

Thus, the bending moment at the center of the beam is only one-half as large as that at the ends of the beam.

The points of inflection are found by equating the expression for the bending moment, M_x, to zero. Thus

$$M_x = -\tfrac{1}{12}wl^2 + \tfrac{1}{2}wlx - \frac{wx^2}{2} = 0,$$

whence

$$x = \tfrac{1}{2}l(1 \pm \tfrac{1}{3}\sqrt{3}).$$

The bending moment diagram is shown in Fig. 131(*d*).

Maximum Deflection.—The expression for the maximum deflection may be found from the elastic curve equation; if the axes are chosen as in Fig. 131(*c*), the elastic curve equation is,

$$EI\frac{d^2y}{dx^2} = M_x = -\tfrac{1}{12}wl^2 + \tfrac{1}{2}wlx - \frac{wx^2}{2}.$$

By integrating this equation twice, determining constants of integration, etc., the value of the maximum deflection is found to be,

$$\Delta = -\frac{1}{384}\frac{wl^4}{EI},$$

which is only one-fifth as large as that of a similarly loaded simple beam (see Art. 46).

It is sometimes convenient to consider the beam in question to be made up of two cantilevers and a simple beam as indicated in Fig. 131(*e*). The shears V' at the points of inflection are

$$V' = V_1 - wx$$
$$= \tfrac{1}{2}wl - w\tfrac{1}{2}l(1 - \tfrac{1}{3}\sqrt{3}).$$

PROBLEMS

138. Select a steel I-beam for a span of 12 ft. if the beam is to be fixed at both ends and subjected to a uniformly distributed load of 200 lb. per ft. Use a working unit-stress of 18,000 lb. per sq. in.

Ans. 3-in. $5\tfrac{1}{2}$-lb. I-beam.

139. Compare the maximum deflections of a timber beam having both ends fixed and a steel beam having simply supported ends. The beams are subjected to the same uniformly distributed load, and have the same span and dimensions.

140. A beam with a clear span of 16 ft. is partially fixed at each end, the negative moment at each end being $-\frac{1}{15}wl^2$. The load is uniformly distributed and equal to 1000 lb. per ft., including the weight of the beam. Find the size of the lightest steel I-beam that may be used without having the fiber unit-stress exceed 16,000 lb. per sq. in.

141. One of the beams in a theater balcony is 18 feet long. The left end is built into the wall in such a manner that the restraint produces a negative bending moment, M_0, at that end equal to 12,000 ft.-lb., and the beam rests on a support 3 feet from the right end. The beam is loaded with a uniformly distributed load of 600 lb. per ft. Determine the reactions on the beam, construct a shear diagram, and determine the value of the maximum positive bending moment.

54. Beam Fixed at Both Ends; Concentrated Load at Mid-span.

—Fig. 132(a) represents such a beam, the length of span being l and the load P. The reactions of the wall may be replaced by the shears V_1 and V_2 and the moments M_1 and M_2 (Fig. 132b) as was done in the preceding articles. These four unknown reactions may be found by use of the conditions of symmetry and equilibrium, and of the equation of the elastic curve of the left half of the beam. Thus, from the conditions of symmetry and equilibrium we have,

$$M_1 = M_2 \quad \text{and} \quad V_1 = V_2 = \tfrac{1}{2}P.$$

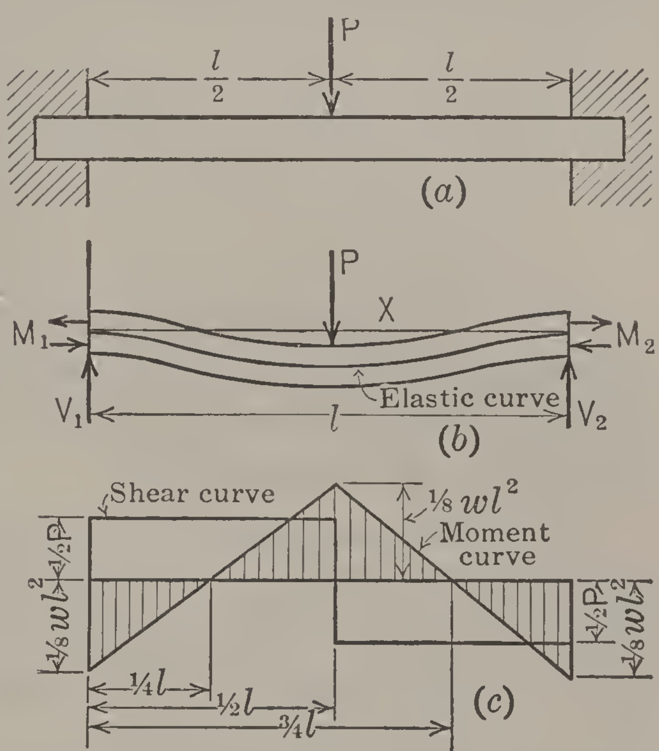


FIG. 132.—Beam fixed at both ends: concentrated load at mid-span.

Now M_1 may be found from the elastic curve equation; if axes are chosen as shown in Fig. 132(b) the equation of the elastic curve of the left half of the beam is

$$EI\frac{d^2y}{dx^2} = M_1 + V_1x = M_1 + \tfrac{1}{2}Px. \quad . \quad . \quad . \quad . \quad (99)$$

Integrating, we have

$$EI\frac{dy}{dx}=M_1x+\frac{Px^2}{4}+c_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

$$\frac{dy}{dx}=0 \text{ when } x=0, \text{ and hence } c_1=0.$$

$$\frac{dy}{dx}=0 \text{ when } x=\frac{l}{2}, \text{ and hence,}$$

$$0=M_1\frac{l}{2}+\frac{Pl^2}{16}.$$

Therefore

$$M_1=-\frac{1}{8}Pl.$$

Thus the bending moment at each end of the beam is negative and equal to $\frac{1}{8}Pl$.

The maximum positive moment occurs at the center where the vertical shear is zero (see Fig. 132*c* for shear diagram) and is

$$\text{Maximum Positive Moment}=M_1+V_1\frac{l}{2}=-\tfrac{1}{8}wl^2+\tfrac{1}{2}P\tfrac{1}{2}l$$
$$=\frac{1}{8}Pl.$$

Thus the maximum positive moment is numerically equal to the maximum negative moment, and hence the maximum fiber unit-stresses at the wall and mid-span sections are equal, if the beam has a constant cross-section.

The inflection point in the curve of the left half of the beam is found, by equating the expression for the bending moment $(-\frac{1}{8}Pl+\frac{1}{2}Px)$ to zero, to be half way between the wall and the center of the span $(x=\frac{1}{4}l)$. The bending moment diagram is shown in Fig. 132(*c*).

By integrating equation (100) and observing that $y=0$ when $x=0$, and that $y=\Delta$ when $x=\frac{l}{2}$, the maximum deflection is found to be

$$\Delta=-\frac{1}{192}\frac{Pl^3}{EI},$$

which is only one-fourth as much as that of a simply supported beam similarly loaded (Art. 47).

PROBLEMS

142. A timber beam 6 in. wide and 12 in. deep is fixed at both ends and has a span of 16 ft. If it is subjected to a concentrated load at the center, what load can it safely resist if a working unit-stress of 800 lb. per sq. in. is used? *Ans.* $P=4800$ lb.

143. Find the maximum deflection of the beam described in the preceding problem when the beam is subjected to a maximum fiber unit-stress of 800 lb. per sq. in.

55. Beam Fixed at One End Supported at Other End, Concentrated Load at Mid-span.—Fig. 133(a) represents such a beam, the length of the span being l and the load, P. As in the preceding articles, the reactions exerted by the wall may be replaced by the shear V_2 and the moment M_2 acting on the section at the wall as shown in Fig. 133(b). Thus there are three unknown reactions (R_1, V_2, M_2) to be found. The left reaction R_1 may be found by use of the equations of the two elastic curves; the expressions for the bending moments on opposite sides of the load are different, and hence the elastic curves on opposite sides of the load are different; but the two curves have a common tangent and a common ordinate under the load. After R_1 has been found from the elastic curve equations, the two equations of equilibrium may be used to find the values of V_2 and M_2.

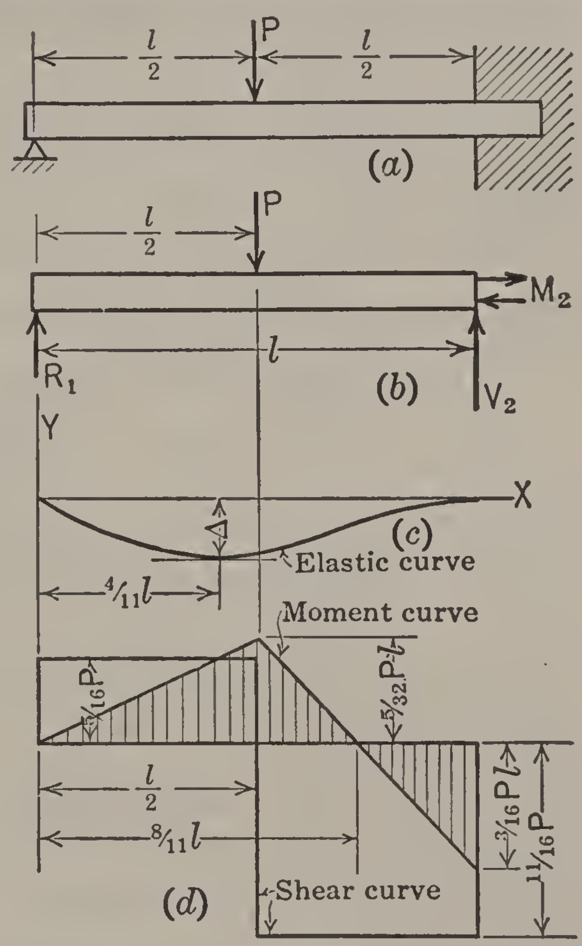


FIG. 133.—Beam fixed at one end, supported at other end: concentrated load at mid-span.

The procedure is as follows: Let the axes be chosen as shown in Fig. 133(c). The equations for the elastic curve of the left half of the beam are:

$$EI\frac{d^2y}{dx^2}=R_1x. \qquad (101)$$

$$EI\frac{dy}{dx}=\tfrac{1}{2}R_1x^2+c_1. \qquad (102)$$

$$EIy=\tfrac{1}{6}R_1x^3+c_1x+c_3. \qquad (103)$$

The similar equations of the elastic curve of the right half are:

$$EI\frac{d^2y}{dx^2}=R_1x-P(x-\tfrac{1}{2}l). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

$$EI\frac{dy}{dx}=\tfrac{1}{2}R_1x^2-\tfrac{1}{2}Px^2+\tfrac{1}{2}Plx+c_2. \quad . \quad . \quad . \quad . \quad (105)$$

$$EIy=\tfrac{1}{6}R_1x^3-\tfrac{1}{6}Px^3+\tfrac{1}{4}Plx^2+c_2x+c_4. \quad . \quad . \quad (106)$$

The five unknowns (c_1, c_3, c_2, c_4 and R_1) may be found by making use of the following conditions: In (103) $y=0$ when $x=0$; in (105) $\frac{dy}{dx}=0$ when $x=l$; $\frac{dy}{dx}$ in (102) $=\frac{dy}{dx}$ in (105), when $x=\frac{l}{2}$ in both (102) and (105); y in (103) $=y$ in (106), when $x=\frac{l}{2}$ in both (103) and (106); and, in (106) $y=0$ when $x=l$.

After the values of the constants are substituted in (103) and (106) the following equations of the elastic curves are found.

$$24EIy=4R_1x^3+3Pl^2x-12R_1l^2x. \quad . \quad . \quad . \quad . \quad . \quad . \quad (103a)$$

$$48EIy=8R_1x^3-8Px^3+12Plx^2-24R_1l^2x+Pl^3, \quad . \quad (106a)$$

and by using the last of the conditions stated above, namely, $y=0$ in (106) when $x=l$, the value of R_1 is found to be

$$R_1=\tfrac{5}{16}P.$$

Now from the two equations of equilibrium we have (see Fig. 133b),

$$\Sigma F=\tfrac{5}{16}P+V_2-P=0.$$

Hence,

$$V_2=\tfrac{11}{16}P.$$

$$\Sigma M_B=\tfrac{5}{16}Pl-P\tfrac{1}{2}-M_2=0.$$

Hence,

$$M_2=-\tfrac{3}{16}Pl.$$

The maximum positive moment occurs at the mid-span and is,

$$\text{Maximum Positive Moment}=\tfrac{5}{16}P\times\tfrac{1}{2}l=\tfrac{5}{32}Pl.$$

The shear and moment diagrams are shown in Fig. 133(d).

56. Comparison with Simple Beams.—In the preceding articles it has been shown that a fixed-ended beam is stronger and stiffer than a similar beam that is simply supported at its ends. This is true because the positive bending moment in the central portion of a beam is decreased by the amount of the negative moment at the ends, and thus more of the material of a fixed beam is effective in resisting the loads than in the simply supported beam, since the bending moment is distributed more nearly uniformly along the length of the fixed beam.

The effect of applying negative moments at the ends of a simply supported beam is shown in Fig. 134. The curve CED, referred to CD as a base line, represents the bending moment diagram for a simply supported beam subjected to a uniformly distributed load of w lb. per ft., the maximum bending moment (as shown in Art. 36, Prob. 88) is at the mid-span and is equal to $\frac{1}{8}wl^2$. Now, if the beam were a fixed-ended beam the negative moments at the ends would be $\frac{1}{12}wl^2$ (as shown in Art. 53) and the curve CED would still represent the moment diagram if AB, Fig. 134(b), instead of CD, represents the axis or base line. Thus the moment at the center is decreased by the amount of the negative moment at the end, and further, *the sum of the maximum positive and negative moments in a beam fixed at both ends is equal to the maximum moment in a similarly loaded simple beam.* The student should show that this statement is also true if the load is concentrated at the mid-span.

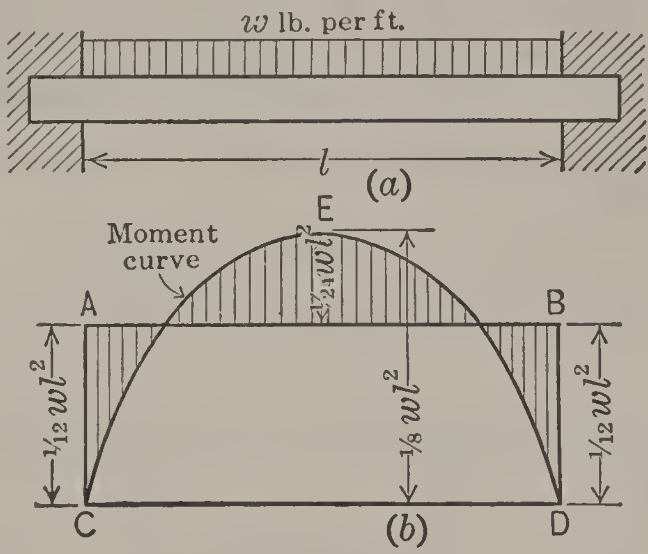


FIG. 134.—Relation between moment diagrams for fixed and simple beams.

In practice, however, it frequently is difficult to determine to what extent a beam is restrained at the ends, since yielding of the end-restraining bodies such as abutments, riveted end-connections, etc., decreases the negative moments at the ends and increases the positive moment by an equal amount. Likewise, unequal settlements of the ends and temperature changes influence the stresses in fixed-ended beams to a greater extent than in simply supported beams. For these reasons fixed-ended beams are not used in preference to simple beams as extensively as the above results would seem to justify. However, test results [2] obtained with certain types of riveted end-connections show that

[2] Tests to Determine the Rigidity of Riveted Joints of Steel Structures. Bull. No. 104, Engineering Experiment Station, University of Illinois.

the moments at the ends are reduced but little by the yielding of these riveted end-restraining connections.

57. Continuous Beams. Theorem of Three Moments.—A continuous beam is one that rests on more than two supports. Since the number of reactions is greater than two, use is made of the elastic curve equation in order to obtain the reactions of the supports, the two equations of equilibrium alone being insufficient.

However, instead of determining the reactions first and then finding the bending moment (and hence fiber unit-stress) at any section, it is more convenient to determine first the negative moments over all the supports and then to find the reactions of the supports and the bending moments and vertical shears at other sections from these negative moments. The negative moments may be found by use of the theorem of three moments which may be derived from the elastic curve equations of two adjacent spans as follows:

Theorem of Three Moments, Load Uniformly Distributed.—Fig. 135(a) represents a continuous beam of several unequal spans;

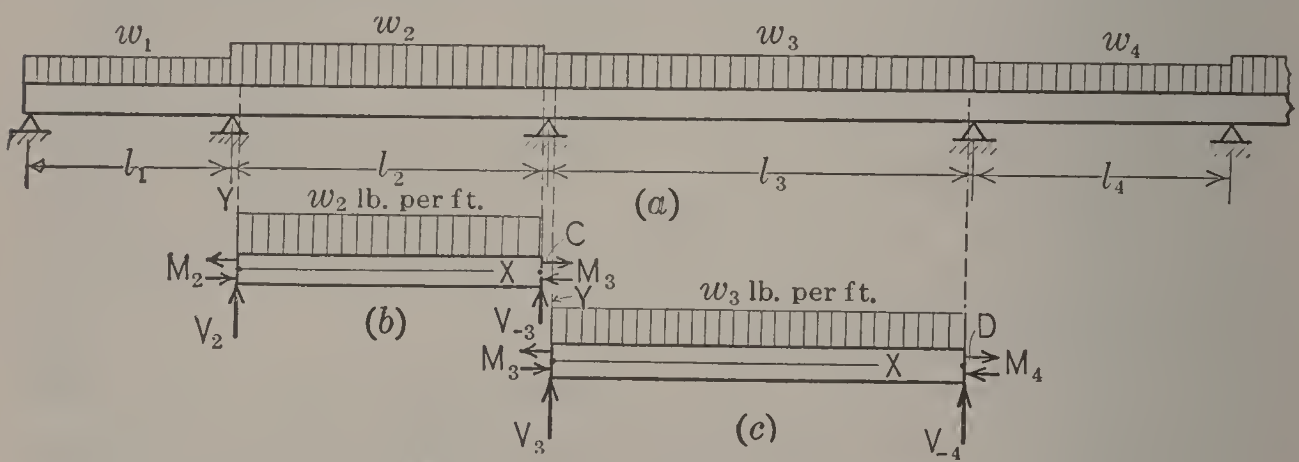


Fig. 135.—Moments at supports of continuous beam.

each span is subjected to a uniformly distributed load, but the load is not constant over the whole length of the beam; all supports are assumed to be on the same level. Consider two adjacent spans (second and third spans) in Fig. 135. Let M_2, M_3, M_4, etc., denote the bending moments at the second, third, fourth, etc., supports. Let V_2 denote the shear on a section just to the right of the second support and V_{-2} just to the left of the second support; similarly let V_3 and V_{-3} denote the shears on the right and left, respectively, of the third support, etc.

A free-body diagram showing all the forces acting on that part of the beam between the second and third supports is shown in Fig. 135(b) and a similar diagram for the part of the beam between the third and fourth supports is shown in Fig. 135(c).

For the second span let the axes be chosen as indicated in Fig. 135(b). Then, the elastic curve equation for the second span is,

$$EI\frac{d^2y}{dx^2}=M_2+V_2x-\frac{w_2x^2}{2}. \qquad (107)$$

$$EI\frac{dy}{dx}=M_2x+\frac{V_2x^2}{2}-\frac{w_2x^3}{6}+c_1. \qquad (108)$$

$$EIy=\frac{M_2x^2}{2}+\frac{V_2x^3}{6}-\frac{w_2x^4}{24}+c_1x+c_2. \qquad (109)$$

Similarly for the third span let the origin be chosen at the second support as shown in Fig. 135 (c). Then

$$EI\frac{d^2y}{dx^2}=M_3+V_3x-\frac{w_3x^2}{2}. \qquad (110)$$

$$EI\frac{dy}{dx}=M_3x+\frac{V_3x^2}{2}-\frac{w_3x^3}{6}+c_3. \qquad (111)$$

$$EIy=\frac{M_3x^2}{2}+\frac{V_3x^3}{6}-\frac{w_3x^4}{24}+c_3x+c_4. \qquad (112)$$

Now in (109), $y=0$ when $x=0$, and hence $c_2=0$; also $y=0$ when $x=\frac{l}{2}$ and hence the value of c_1 may easily be obtained. Again, in (112) $y=0$ when $x=0$, and hence $c_4=0$; also $y=0$ when $x=l_3$ and hence the value of c_3 may be obtained. Further $\frac{dy}{dx}$ in (108) is equal to $\frac{dy}{dx}$ in (111) when x in (108) is equal to l_2 and x in (111) is equal to zero. By making use of these facts the following equation is found.

$$12M_2l_2+8V_2l_2^2-3w_2l_2^3=-12M_3l_3-4V_3l_3^2+w_3l_3^3. \qquad (113)$$

Now V_2 and V_3 may be expressed in terms of M_2, M_3, M_4 and the known quantities l_2, l_3, w_2 and w_3 by using one of the equations of equilibrium for the forces in Fig. 135(b) and in Fig. 135(c). Thus, for the second span (Fig. 135b) one equilibrium equation is

$$\Sigma MC=M_2+V_2l_2-\tfrac{1}{2}w_2l_2^2+M_3=0.$$

Hence,

$$V_2=\frac{-M_2+\frac{1}{2}w_2l_2^2-M_3}{l} \qquad (114)$$

and for the third span,

$$\left.\begin{aligned} &\Sigma M_D = M_3 + V_3 l - \tfrac{1}{2} w_3 l_3^2 + M_4 = 0. \\ &\text{Hence,} \quad V_3 = \frac{-M_3 + \tfrac{1}{2} w_3 l_3^2 - M_4}{l} \end{aligned}\right\}, \quad . \quad . \quad . \quad (115)$$

in which all the moments are assumed to be positive.

By substituting these values of V_2 and V_3 in equation (113) the following relation between the bending moments at three consecutive supports is found:

$$M_2 l_2 + 2M_3(l_2 + l_3) + M_4 l_3 = -\tfrac{1}{4} w_2 l_2^3 - \tfrac{1}{4} w_3 l_3^3,$$

or, if the subscripts 1, 2, and 3 are used to refer to any three consecutive supports and spans the equation may be written,

$$M_1 l_1 + 2M_2(l_1 + l_2) + M_3 l_2 = -\tfrac{1}{4} w_1 l_1^3 - \tfrac{1}{4} w_2 l_2^3. \quad . \quad (116)$$

which expresses the *theorem of three moments* for a continuous beam subjected to uniformly distributed loads and resting on supports on the same level.

If the spans are equal $(l_1 = l_2 = l)$ and each span carries the same load $(w_2 = w_3 = w)$ the equation becomes

$$M_1 + 4M_2 + M_3 = -\tfrac{1}{2} w l^2, \quad . \quad . \quad . \quad . \quad (117)$$

which expresses the theorem of three moments for a continuous beam with equal spans and with a constant uniformly distributed load over the entire beam.

58. Solution of Typical Problem.—Let it be required to draw the shear and moment diagrams for a continuous beam of four spans, each of length l, when subjected to a load of w lb. per ft. over the entire beam, and also to find the maximum fiber unit-stress and the equation of the elastic curve of the first span. (See Fig. 136(a).) The known quantities are: w, l, E and I. The main steps in the solution are:

(a) By use of the theorem of three moments find the negative bending moment over all supports.

(b) Find the reactions of the supports from the negative moments over the supports; all the external forces acting on the beam will then be known.

(c) Find the vertical shears at various sections and draw the shear diagram.

(*d*) Find the bending moments at various sections and draw the moment diagram.

(*e*) Equate the maximum bending moment to $\frac{sI}{c}$ and solve for s, the maximum fiber unit-stress.

(*f*) In the elastic curve equation $EI\frac{d^2y}{dx^2}=M$ substitute for M the expression for the bending moment in the first span and integrate twice, etc.

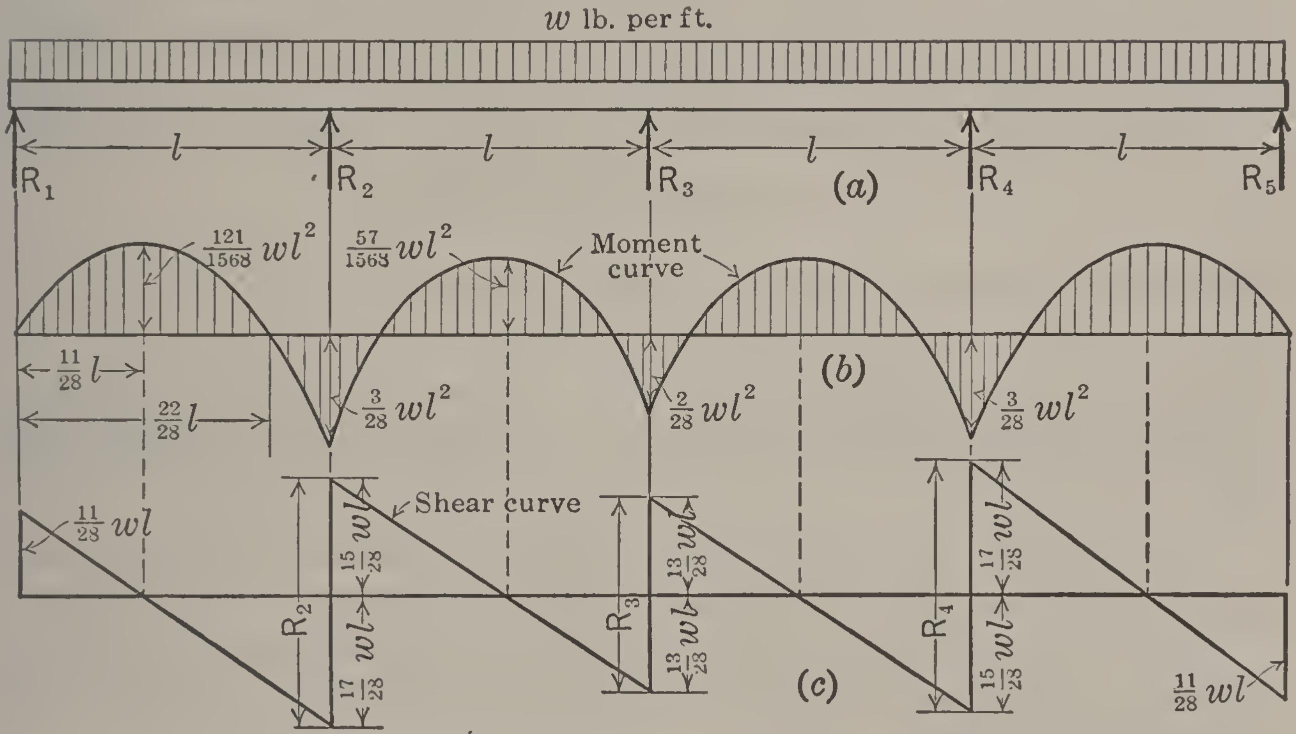


FIG. 136.—Moment and shear diagrams for continuous beam.

Solution.—(*a*) In accordance with the theorem of three moments, the following equations may be written:

$$M_1+4M_2+M_3=-\tfrac{1}{2}wl^2,$$

$$M_2+4M_3+M_4=-\tfrac{1}{2}wl^2,$$

$$M_3+4M_4+M_5=-\tfrac{1}{2}wl^2.$$

Further, $M_1=M_5=0$ since there is no restraint at the ends; and $M_2=M_4$, from the condition of symmetry. The solution of these equations gives the following values:

$$M_1=0,\ M_2=-\tfrac{3}{28}wl^2,\ M_3=-\tfrac{2}{28}wl^2,\ M_4=-\tfrac{3}{28}wl^2,\ M_5=0.$$

(*b*) Now the bending moment at a section over the second support as found above is $-\frac{3}{28}wl^2$, but the bending moment at any section is the algebraic sum of the moments of the forces to the left of the section. Hence,

$$R_1l-wl\frac{l}{2}=-\tfrac{3}{28}wl^2;\ R_1=\tfrac{11}{28}wl.$$

Similarly for the moment over the third support we have,

$$R_1 \cdot 2l + R_2 l - w \cdot 2l \cdot l = -\tfrac{2}{28}wl; \quad R_2 = \tfrac{32}{28}wl.$$

In a similar way the value of R_3 is found to be, $R_3 = \frac{26}{28}wl$, and from symmetry, $R_4 = R_2$ and $R_5 = R_1$.

(*c*) The shear and moment diagrams are shown in Fig. 136(*b*) and (*c*). The vertical shear at any section in the first span at the distance x from the left support is

$$V_x = R_1 - wx = \tfrac{11}{28}wl - wx,$$

and hence the vertical shear is zero when $x = \frac{11}{28}l$; the maximum positive moment occurs at this section.

The shear just to the left of the second support is

$$V_{-2} = R_1 - wl = \tfrac{11}{28}wl - wl = -\tfrac{17}{28}wl,$$

and the shear just to the right of the second support is

$$V_2 = R_1 + R_2 - wl = \tfrac{11}{28}wl + \tfrac{32}{28}wl - wl = +\tfrac{15}{28}wl.$$

Thus, the shear changes at the second support from $-\frac{17}{28}wl$ to $+\frac{15}{28}wl$ due to the reaction R_2; in other words, the reaction of a support is equal to the arithmetic sum of the shears at the two sides of the support.

The shears in other spans may be found by the same method as used above for the first span; the values are given in Fig. 136(*b*).

Thus the reaction at any support may be found by solving for the shears on the two sides of the support and adding them rather than by the method used under (*b*) above.

(*d*) The bending moment at any section in the first span is

$$M'_x = R_1 x - \frac{wx^2}{2},$$

and as noted above this is maximum when $x = \frac{11}{28}l$. Therefore, the maximum positive moment is

$$\text{Max. pos. moment} = \frac{11}{28}wl \cdot \frac{11}{28}l - \frac{w}{2}\left(\frac{11}{28}l\right)^2 = \frac{121}{1568}wl^2,$$

which is less than the maximum negative moment $(\frac{3}{28}wl^2)$.

The inflection point occurs where $M'_x = 0$; thus

$$M'_x = \frac{11}{28}wl \cdot x - \frac{wx^2}{2} = 0,$$

whence

$$x = \tfrac{22}{28}l.$$

The bending moment at any section in the second span is

$$M''_x = M_2 + V_2 x - wx^2,$$

$$= -\tfrac{3}{28}wl^2 + \tfrac{15}{28}wlx - wx^2.$$

The moment is maximum when $x=\frac{15}{28}l$, since this is the value that makes the shear in the second span equal to zero. The value of the maximum positive moment in this span is $\frac{57}{1568}wl^2$.

(*e*) The greatest bending moment that the beam is subjected to is the negative moment at the second (or fourth) support, and hence the maximum unit-stress in the beam is

$$s=\frac{Mc}{I}=\frac{3}{28}wl^2\cdot\frac{c}{I}.$$

(*f*) The elastic curve equation for the first span is

$$EI\frac{d^2y}{dx^2}=R_1x-\frac{wx^2}{2}=\frac{11}{28}wlx-\frac{wx^2}{2}.$$

Integrating we have,

$$EI\frac{dy}{dx}=\frac{11}{28}wl\frac{x^2}{2}-\frac{wx^3}{6}+c_1$$

$$EIy=\frac{11}{28}wl\frac{x^3}{6}-\frac{wx^4}{24}+c_1x+c_2.$$

Now, when $x=0$, $y=0$; and when $x=l$, $y=0$; thus $c_2=0$, and c_1 may easily be found to be $-\frac{1}{42}wl^3$. Therefore the elastic curve equation of the first span is

$$EIy=\frac{11}{28}wl\frac{x^3}{6}-\frac{wx^4}{24}-\frac{1}{42}wl^3x.$$

59. Values of Moments and Shears.—In Fig. 137 are given the values of the coefficients from which the negative moments

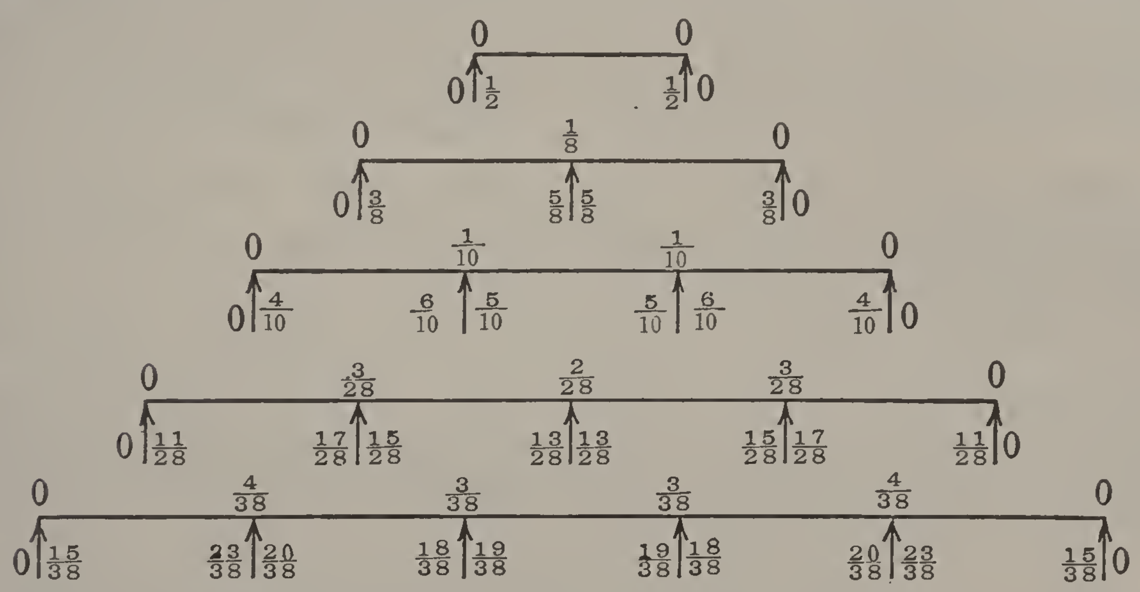


FIG. 137.—Values of shear and moment coefficients for continuous beam of equal spans subjected to uniform load.

over the supports and the vertical shears at either side of each support may be found for continuous beams resting on supports on the same level, having equal spans of l ft. each, and carrying a constant uniform load of w lb. per ft. on each span.

The values directly above the supports are the moment coefficients, that is, the numbers by which $-wl^2$ must be multiplied to obtain the bending moments over the supports. Similarly, the numbers on either side of the supports are the shear coefficients, that is, the numbers by which wl must be multiplied to obtain the magnitude of the vertical shear for the section; the vertical shear is negative for the section to the left of each support and positive for the section to the right of each support. The reaction at each support is the arithmetic sum of the vertical shears on the two sides of the support. The values given in Fig. 137 may be found by the methods discussed in Art. 58.

PROBLEMS

144. A continuous beam consists of four equal spans each 12 ft. long, and is subjected to a uniform load of 100 lb. per ft. over its entire length. (*a*) Find the reaction of the second support. (*b*) If the beam has a rectangular cross-section, the depth being twice the width, what is the depth if the maximum fiber unit-stress in the beam is 100 lb. per sq. in.?

Ans. (*a*) $R_2 = 1370$ lb., (*b*) depth $= 6.06$ in.

145. A standard 12-in. 40-lb. I-beam is used as a horizontal continuous beam over four spans. The lengths of the spans are 12, 16, 16 and 12 ft. respectively, from left to right. The loads are uniformly distributed over each span, the loads per foot of length on the four spans being 1600, 2400, 2400, and 1600 lb. per ft., respectively. Find (*a*) the maximum fiber unit-stress in the beam. *Ans.* (*a*) $s = 16{,}630$ lb. per sq. in.

146. A continuous beam has three spans of 10 ft. each. The first span is subjected to a uniformly distributed load of 3000 lb. per ft. The other spans are not loaded. Find the four reactions.

Ans. $R_1 = 13{,}000$ lb. $R_2 = 19{,}500$ lb. $R_3 = -3000$ lb. $R_4 = 500$ lb.

147. What should be the depth of a pine continuous beam having four equal spans of 7 ft. in order to resist a uniformly distributed load of 300 lb. per ft. over each span? Use a working unit-stress of 1000 lb. per sq. in. and make the depth of the beam twice the width.

148. A 10-in. 25-lb. I-beam is continuous over three spans, each having a length of 16 ft. The first and third spans are loaded with 800 lb. per ft. and the second span is not loaded. Find the maximum fiber unit-stress in the beam and draw, to scale, the moment and shear diagrams.

149. A beam is continuous over three spans of 20, 10, and 15 ft. respectively. The first and third spans are each loaded by a uniformly distributed load of 400 lb. per ft. and the second span is not loaded. Find the moment over each support and the shear at the right of the first support.

60. Advantages and Disadvantages of Continuous Beams.—The remarks in Art. 56 concerning fixed-ended beams apply also, in the main, to continuous beams; the negative moments over the supports reduce the moments (and hence stresses) near the centers of the spans, and hence continuous beams are stronger and stiffer than simply supported beams of equal spans. On the other hand, the uneven settlement of supports may change the moments at the supports and throughout the beam from those found by the analysis given above; further, the stiffer the beams the greater is the change in the moments. Partly for this reason and partly because the loads on the various spans of continuous beams in structures may vary considerably from that assumed in the design, it is frequently assumed in practice that in uniformly loaded beams of four or more equal spans all the spans (except the end spans), are subjected to the same maximum bending moment, a common value being $\frac{1}{12}wl^2$; the maximum moment in the end spans is somewhat greater ($\frac{1}{10}wl^2$ is frequently used). Compare these values with those given in Fig. 137 for beams having more than four spans and note that the moments do not vary greatly over the inner supports. Two-span and three-span beams are stressed higher than beams of four or more spans, and for such beams the maximum moment in each span is frequently assumed to be $\frac{1}{10}wl^2$. However, the values used in any case may be based on the results of an analysis of the moments in continuous beams similar to that discussed in the preceding articles.

61. Theorem of Three Moments for Concentrated Loads.—The loading on most continuous beams may be assumed to be uniformly distributed without introducing serious errors, but in some cases heavy concentrated loads act on the beams and the results given in Art. 59 are not applicable. Space does not permit the derivation, but the theorem of three moments that applies to a beam of constant cross-section with supports on the same level and subjected to concentrated loads is as follows:

$$M_1l_1+2M_2(l_1+l_2)+M_3l_3$$
$$=-P_1l_1^2(k-k^3)-P_2l_2^2(2k-3k^2+k^3). \quad (118)$$

M_1, M_2 and M_3 are the bending moments at any three consecutive supports, l_1 and l_2 being the lengths of the two consecutive spans at the ends of which the moments are M_1, M_2 and M_3.

P_1 is a concentrated load in the first of the two spans at the distance kl_1 from the first of the three supports, and P_2 a concentrated load in the second of the two spans at the distance kl_2 from the second of the three supports, where k_1 and k_2 are proportional parts of the spans such as $\frac{1}{3}$, $\frac{1}{2}$, etc.

PROBLEM

150. A continuous beam has two spans, the left span 20 ft. long and the right span 18 ft. long. The 20-ft. span has a load of 1600 lb. 10 ft. from the left end, and the 18-ft. span has a load of 15,000 lb. 8 ft. from the right end. Find the proper size of a steel I-beam to resist these loads using a working unit-stress of 16,000 lb. per sq. in. *Ans.* 12-in. 40-lb. I-beam.

CHAPTER VIII

DEFLECTION OF STATICALLY DETERMINATE BEAMS

(Moment-area Method)

Note. Arts. 44 to 46 should be studied before this chapter is read.

62. Introduction.—In Chapters VI and VII the elastic curve equation $M = \pm EI\dfrac{d^2y}{dx^2}$ was used (1) to find the deflection of beams and (2) to find the reactions (both forces and couples) of beams for which the number of unknown reactions was greater than the number of equilibrium equations, that is, of statically indeterminate beams. The method of treating the equation in each case was to express M in terms of the reactions and x, and then to integrate both sides of the equation, determining constants in the equations (whether constants of integration or reactions) from the physical conditions satisfied by the beam. This method was called the double-integration method.

Another method of treating the equation leads to the use of moments of areas of the moment-diagram $\left(\text{or of the } \dfrac{M}{EI}\text{-diagram}\right)$ for finding deflections of beams and reactions of statically indeterminate beams, and hence the method is called the *moment-area* method. For various purposes and for certain types of beams, as for example beams having a variable moment of inertia, it possesses advantages over the double integration method although it should, perhaps, be considered supplementary to, rather than a substitute for, the double integration method.

Two methods of using moments of areas of the moment-diagram $\left(\text{or of the } \dfrac{M}{EI}\text{-diagram}\right)$ will be discussed briefly; namely, the *slope-deviation* method and the *conjugate beam* method.[1]

[1] See " Deflections of Beams by the Conjugate Beam Method," by H. M. Westergaard, Journal of the Western Society of Engineers, vol. 26, Nov., 1921.

These two methods will be used in this chapter for determining the deflection of statically determinate beams, and the same two methods will be used in the following chapter for determining the reactions, as well as the deflections, of statically indeterminate beams.

Slope Deviation Method

63. Theorems of the Slope-deviation Method.—As explained in Art. 45, the equation of the elastic curve of a beam is $M = -EI\frac{d^2y}{dx^2}$, in which M is positive when it produces tensile stress on the bottom of a horizontal beam, and y is positive when measured downwards. This equation may be written

$$-\frac{M}{EI} = \frac{d\left(\frac{dy}{dx}\right)}{dx} \quad \text{or} \quad -\frac{M}{EI}dx = d\left(\frac{dy}{dx}\right), \quad . \quad . \quad (119)$$

in which $\frac{dy}{dx}$ is the slope of the elastic curve at a point for which the bending moment is M. If θ denotes the angle made by the tangent line to the elastic curve with the axis of the undeflected beam, then $\frac{dy}{dx} = \tan\theta$. But in a well designed beam θ is small, and hence, without appreciable error, $\tan\theta$ may be replaced by θ. Therefore,

$$d\theta = \frac{M}{EI}dx \quad \text{and} \quad \Delta\theta = \int \frac{M}{EI}dx, \quad . \quad . \quad . \quad (120)$$

in which $\Delta\theta$ denotes the change or increment in the angle θ; θ is considered positive when measured in the clockwise direction of rotation.

A graphical interpretation of this equation may be made as follows: Fig. 138(a) represents a beam subjected to a distributed load; Fig. 138(b) represents the $\frac{M}{EI}$-diagram for the beam, any ordinate in which is the bending moment at the section where the ordinate is erected divided by EI; if the beam is homogeneous and has a constant cross-section, EI is a constant and hence the $\frac{M}{EI}$-diagram has the same form as the moment-diagram. In Fig. 138(c) $PABQ$ represents the elastic curve of the beam.

Now, as indicated in Fig. 138(b), $\frac{M}{EI}dx$ is represented by an elementary area under the $\frac{M}{EI}$-diagram and $\int_A^B \frac{M}{EI}dx$ is represented by the total area under the $\frac{M}{EI}$-diagram between specified ordinates, such as ordinates erected at A' and B'. Further, $d\theta$ represents, as shown in Fig. 138(c), the change in the slope of the elastic curve at two points a short distance, dx, apart, and hence $\int_{\theta_B}^{\theta_A} d\,\theta = \Delta\theta$ represents the change in slopes of the elastic curve at any two points, such as A and B. Therefore, equation 120 leads to the following theorem:

Theorem I. When a straight beam is subjected to bending the difference in the slopes of the elastic curve at any two points is equal in magnitude to the area of the $\frac{M}{EI}$*-diagram between the two points.*

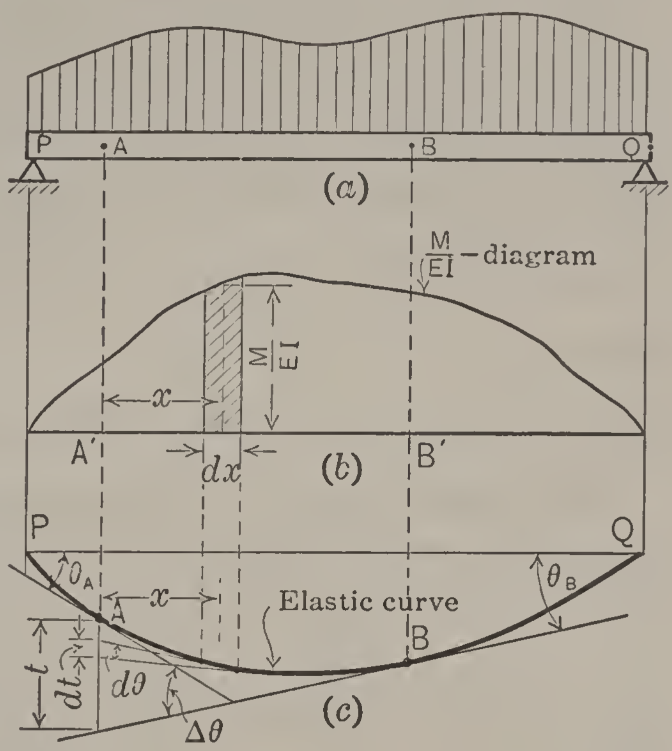


FIG. 138.—Deflection of a beam; slope-deviation method.

Now let t, Fig. 138(c), denote the distance of any point A on the elastic curve, measured in a direction perpendicular to the original position of the beam, from a tangent drawn at any other point B on the elastic curve. The distance t will be called the *tangential deviation* of the point. Then, from Fig. 138(c),

$$t = \int dt = \int x d\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (121)$$

and from equation 120,

$$d\theta = \frac{M}{EI}dx.$$

Therefore,

$$t = \int \frac{Mx}{EI}dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

But, as is evident from Fig. 138(b), $\frac{Mx}{EI}dx$ is the moment of the elementary area, $\frac{Mdx}{EI}$, of the $\frac{M}{EI}$-diagram about the ordinate through the point A whose tangential deviation is t, and hence $\int \frac{Mx}{EI}dx$ is the moment of that part of the $\frac{M}{EI}$-diagram between the two ordinates considered; the moment being taken about the ordinate through the point whose tangential deviation is desired. The following theorem therefore may be stated:

Theorem II.—When a straight beam is subjected to bending the distance of any point A on the elastic curve, measured normal to the original position of the beam, from a tangent drawn to the elastic curve at any other point B, is equal in magnitude to the moment of the area of the $\frac{M}{EI}$*-diagram between the two points about an ordinate through A.*

Applications

64. Simple Beam. Concentrated Load at Mid-span.—The beam is shown in Fig. 139(a); the weight of the beam is assumed to be negligible. Fig. 139(b) shows the $\frac{M}{EI}$-diagram; the dangerous section is at the mid-span; the maximum bending moment is $\frac{Pl}{4}$ (Art. 36), and the bending moment at a section the distance x from the left support is $\frac{P}{2}x$. Since the beam is assumed to be homogeneous and to have a constant cross-section, the $\frac{M}{EI}$-diagram has the same form as the moment-diagram. In Fig. 139(c) ACB represents the elastic curve of the beam.

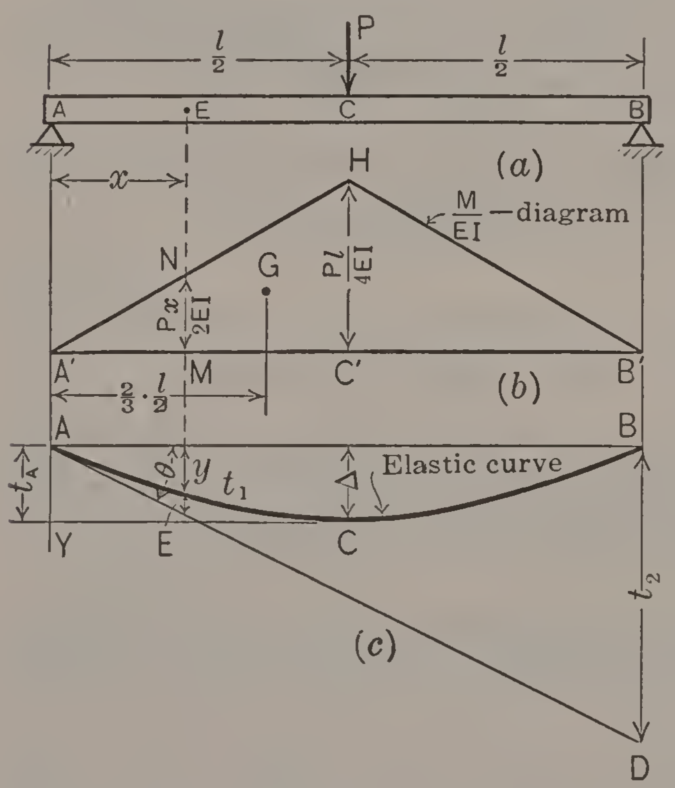


FIG. 139.—Deflection of simple beam, load at center; slope-deviation method.

Let it be required to find the maximum deflection Δ, which occurs at the center of the span. The tangential deviation of the point A from a tangent at C is t_A (Fig. 139c) which is equal in magnitude to Δ, and which may be found from Theorem II of Art. 63. Thus,

$$\begin{aligned}\Delta = t_A &= \text{Moment of area } A'C'H \text{ about } A' \\ &= \text{Area } A'C'H \text{ times distance to centroid, } G, \text{ of area } A'C'H \textit{ from } A' \\ &= \frac{1}{2}\frac{Pl}{4EI}\frac{l}{2}\cdot\frac{2}{3}\frac{l}{2} = \frac{Pl^3}{48EI}.\end{aligned}$$

Next let it be required to find the deflection y of a point E (Fig. 139c) at a distance x from the left support. If a tangent AD is drawn to the elastic curve at A, t_2 is the tangential deviation of B and t_1 is the tangential deviation of the point E. From the geometry of the figure we have,

$$\frac{y+t_1}{t_2} = \frac{x}{l} \quad \text{or} \quad y = t_2\frac{x}{l} - t_1,$$

and since the slopes of the elastic curve are small we have, $\theta = \dfrac{t_2}{l}$. Therefore,

$$y = \theta x - t_1.$$

Now θ is equal to the change of the slopes at A and C and hence, according to Theorem I,

$$\begin{aligned}\theta &= \text{area } A'C'H \\ &= \frac{1}{2}\frac{Pl}{4EI}\frac{l}{2} = \frac{1}{16}\frac{Pl^2}{EI},\end{aligned}$$

and, according to Theorem II,

$$\begin{aligned}t_1 &= \text{moment of area } A'MN \text{ about } MN \\ &= \frac{1}{2}\frac{Px}{EI}\cdot\frac{x}{2}\cdot\frac{x}{3} = \frac{Px^3}{12EI}.\end{aligned}$$

Therefore

$$y = \frac{Pl^2x}{16EI} - \frac{Px^3}{12EI},$$

or

$$48EIy = Px(3l^2 - 4x^2),$$

which is the elastic curve equation for the left half of the beam.

65. Cantilever Beam. Concentrated Load at End.—In Fig. 140(a) AB represents the elastic curve of the beam, and in Fig. 140(b), $A'B'H$ represents the $\dfrac{M}{EI}$-diagram. The beam is assumed to have a constant cross-section and the weight of the beam is neglected. The maximum deflection of the beam may be found as follows.

The tangential deviation, t_A, of A from a tangent drawn at B is equal to the maximum deflection Δ of the beam. Thus from Theorem II we have,

$$\begin{aligned}\Delta = t_A &= \text{moment of area } A'B'H \text{ about } A' \\ &= \text{Area } A'B'H \text{ times distance to centroid.} \\ &= \frac{1}{2}\frac{Pl^2}{EI}\cdot\frac{2}{3}l = \frac{1}{3}\frac{Pl^3}{EI}.\end{aligned}$$

The deflection, y, of a point D at the distance x from the free end may be found as follows: The tangential deviation, t_D, of D from a tangent at B is equal to y; thus, using Theorem II we have,

$$y = t_D = \text{moment of area } B'D'EH \text{ about } D'E.$$

Let the area be divided into two triangular areas as indicated by the dotted line. Then

$$y = t_D = \frac{Px}{2EI}\cdot\frac{(l-x)^2}{3} + \frac{Pl}{2EI}\cdot\frac{2}{3}(l-x)^2.$$

Hence,

$$6EIy = P(x^3 - 3l^2x + 2l^3)$$

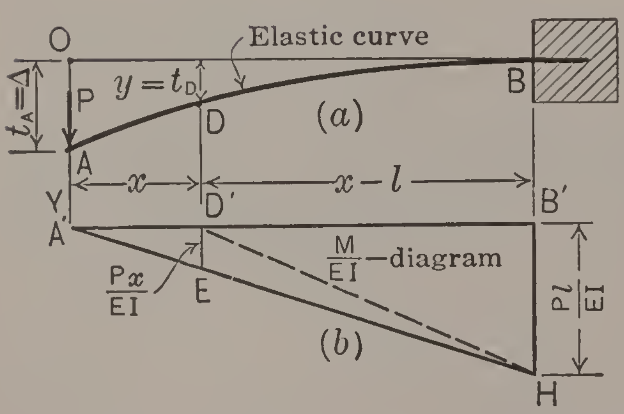


FIG. 140.—Deflection of cantilever beam, load at end; slope-deviation method.

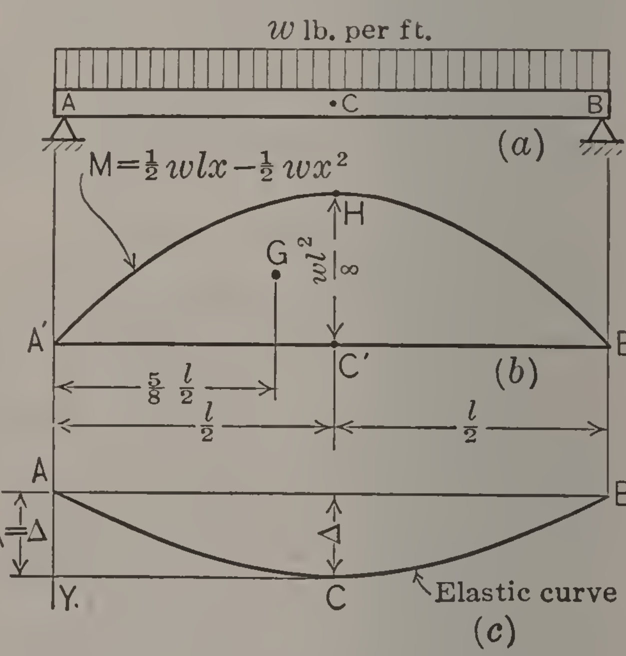


FIG. 141.—Deflection of simple beam; uniform load; slope-deviation method.

which is the elastic curve equation for the beam.

66. Simple Beam; Load Distributed Uniformly.—The beam shown in Fig. 141(a) has a constant cross-section and the weight of the beam is negligible. The moment diagram $A'HB'$ (Fig.

141b) is a parabola, the ordinate M to any point being expressed by the equation $M=\frac{1}{2}wlx-\frac{1}{2}wx^2$, and the maximum ordinate being $\frac{1}{8}wl^2$ (see Prob. 88). The elastic curve is represented by the line ACB in Fig. 141(c).

The maximum deflection of the beam may be found as follows: The tangential deviation t_A of the point A (Fig. 141c) from a tangent at C is equal in magnitude to the maximum deflection, Δ, of the beam. Hence according to Theorem II we have

$$\Delta=t_A=\frac{1}{EI}\text{ (moment of area } A'C'H \text{ about } A').$$

Now,

Area $A'C'H=\frac{2}{3}\overline{A'C'}\times\overline{C'H}$ (see Art. 159, Appendix II), and the distance of the centroid of area $A'C'H$ from A' is $\frac{5}{8}\overline{AC'}$. Therefore,

$$\Delta=t_A=\frac{1}{EI}\left(\frac{2}{3}\cdot\frac{1}{8}wl^2\cdot\frac{l}{2}\cdot\frac{5}{8}\frac{l}{2}\right)$$
$$=\frac{5}{384}\frac{wl^4}{EI}=\frac{5}{384}\frac{Wl^3}{EI}.$$

67. Cantilever Beam; Load Distributed Uniformly.—The elastic curve of the beam is represented by the line AB (Fig. 142a), and the moment-diagram for the beam is shown in Fig. 142(b). The maximum moment, at the wall, is $-\frac{1}{2}wl^2$ and the moment at a distance x from the free end is $-\dfrac{wx^2}{2}$. Now the tangential deviation of the point A from a tangent drawn at B is equal in magnitude to the maximum deflection, Δ. Hence, by integrating equation (122), we have

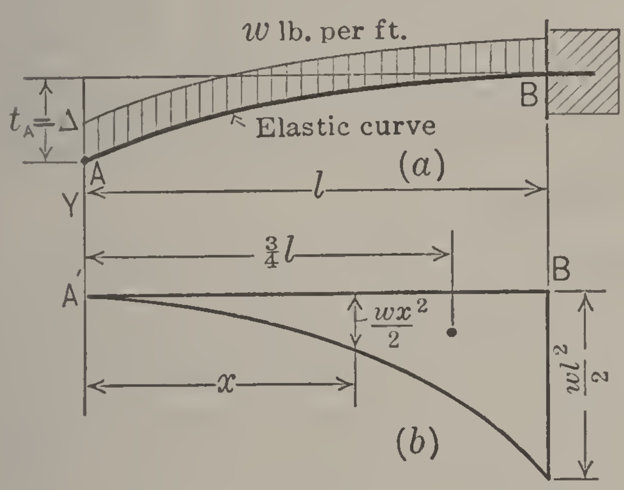


FIG. 142. — Deflection of cantilever beam, uniform load; slope-deviation method.

$$\Delta=t_A=\frac{1}{EI}\int_0^l Mxdx$$
$$=\frac{1}{EI}\int_0^l \frac{wx^2}{2}xdx$$
$$=\frac{wl^4}{8EI}=\frac{1}{8}\frac{Wl^3}{EI},$$

or, from Theorem II, the value of Δ is found without integrating:

$$\Delta = t_A = \frac{1}{EI}\text{ (moment of area } A'B'H \text{ about } A')$$

$$= \frac{1}{EI}\left(\frac{1}{3}\frac{wl^2}{2}\cdot l\cdot\frac{3}{4}l\right), \text{ (See Art. 159, Appendix II)}$$

$$= \frac{1}{8}\frac{wl^4}{EI}.$$

Conjugate-beam Method

68. Conjugate Beam Defined. Equations Stated.—By the conjugate-beam method the deflection at any section of a beam (here called the "given" beam) is found by calculating the bending moment at the corresponding section of another beam (here called the conjugate beam), the conjugate beam being subjected to a distributed load such that the intensity of the load at any section is proportional to the ordinate of the $\frac{M}{EI}$-diagram at that section. That is, the conjugate beam may be assumed to be loaded with sand the depth of which over any section of the beam is, according to a certain scale, the ordinate of the $\frac{M}{EI}$-diagram at that section. To describe this loading the conjugate beam will be said to be loaded with the $\frac{M}{EI}$-diagram.

By definition the conjugate beam is one that (a) has a length, l', which is equal to the length, l, of the given beam, (b) is in equilibrium, and (c) is so loaded that the bending moment M' at any section is equal to the deflection, y, at the corresponding section in the "given" beam. The defining equations then are: From (a)

$$l' = l;$$

from (b), since the loads and reactions constitute a parallel system of vectors,[2] there are two equations of equilibrium; namely,

$$\Sigma F' = 0 \quad \text{and} \quad \Sigma M' = 0$$

and from (c)

$$M'_x = y_x.$$

[2] It will be found that the loads and reactions are not forces; that is, they are not expressed in pounds, tons, etc., but for convenience they may be called *elastic forces*, since they involve the elastic properties of the beam.

The last equation requires, as is shown below, (1) that the vertical shear, V', for any section of the conjugate beam shall be equal to the slope at the corresponding point of the elastic curve of the "given" beam; that is,

$$V'=\frac{dy}{dx}=\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123)$$

and (2) that the intensity, w', of the distributed load on the conjugate beam at any section shall be equal to the $\frac{M}{EI}$ for the corresponding section in the given beam; in other words, the conjugate beam is loaded with the $\frac{M}{EI}$-diagram. Thus,

$$w'=\frac{M}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (124)$$

Proof of Equations (123) *and* (124).—Let a straight beam be subjected to a distributed load only; the intensity of the load at the distance x from the left support is w_x, and the vertical shear and bending moment for this section are V_x and M_x. In the following equations the sign of the bending moment is determined as stated in Art. 34; the vertical shear V_x is positive when directed upwards; the deflection y is positive in the downward direction, and x is positive to the right, and hence the elastic curve equation is $M=-EI\frac{d^2y}{dx^2}$; further, $\theta\left(=\frac{dy}{dx}\right)$ is positive when measured in the clockwise direction of rotation.

Now the difference, dV, in the vertical shears for two sections the distance dx apart is

$$dV=wdx \quad \text{or} \quad w=\frac{dV}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

But from Art. 37,

$$V=\frac{dM}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

and hence,

$$\frac{dV}{dx}=\frac{d^2M}{dx^2}=w. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

Now the elastic curve equation of the beam is (see Art. 45),

$$\frac{d^2y}{dx^2}=\frac{M}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (128)$$

Thus a comparison of equations (127) and (128) shows that the deflection y of the "given" beam would be equal to the bending moment M' at the corresponding section of another beam (the conjugate beam) which has a distributed load equal to $\frac{M}{EI}$; that is, the conjugate beam is loaded with the $\frac{M}{EI}$-diagram.

Further, since the slope θ of the elastic curve at any point is $\theta = \frac{dy}{dx}$, it follows that, if the conjugate beam is loaded so that $M' = y$, then from equation (126)

$$V' = \frac{dy}{dx} = \theta.$$

That is, the vertical shear for any section in the conjugate beam must be equal to the slope of the elastic curve at the corresponding section in the given beam.

It will be noted that, in the conjugate beam method, the actual *elastic* beam is replaced by a *rigid* beam with the elastic properties of the actual beam introduced in the loads and reactions of the rigid beam. For this reason the method is sometimes referred to as *the method of elastic weights.*

Summarizing: The equations, then, that the *conjugate* beam for any *given* beam must satisfy are

$$l' = l, \quad \Sigma F' = 0, \quad \Sigma M' = 0, \qquad M' = y$$

$$V' = \theta, \qquad w' = \frac{M}{EI}.$$

The applications of these equations in finding the deflection of various types of beam is given below.

Applications

69. Simple Beam; Concentrated Load at Mid-span.—The beam is bent as shown in Fig. 143(*a*). The bending moment diagram (M-diagram) is shown in Fig. 143(*b*) and since the beam has a constant cross-section the $\frac{M}{EI}$-diagram will have the same form as the M-diagram. The conjugate beam is shown in Fig. 143(*c*); the bending moment at the mid-span section of the conjugate beam is equal to the deflection of the mid-span section (maximum deflection) of the given beam, provided that the conjugate beam is made to satisfy equations of Art. 68. This may be done by selecting the beam as shown in Fig. 143(c) in which $l'=l$; the distributed load is the $\frac{M}{EI}$-diagram; since $M'=y$, and since $y=0$ at the ends of the " given " beam, the moment M' at the ends of the conjugate beam must also equal zero, and hence the ends of the conjugate beam are subjected to zero moments, that is, the ends are free to turn as shown in Fig. 143(c); further, since $V'=\theta$ and since θ is not zero at the ends of the given beam, there must be vertical shears at the ends of the conjugate beam, and these could be produced by reactions R'_1 and R'_2 of supports at the ends. Thus, if the " given " beam is a simple beam the conjugate beam also is a simple beam.

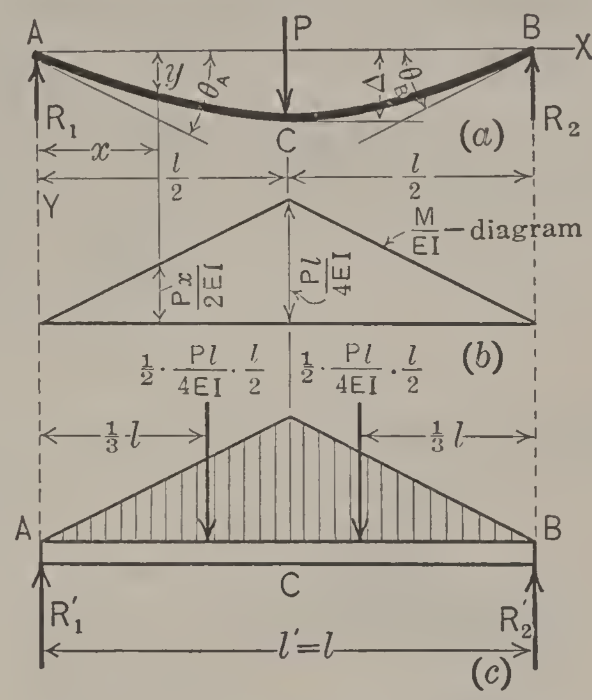


FIG. 143. — Deflection of simple beam, load at center; conjugate-beam method.

Now the maximum deflection Δ of the given beam occurs at the mid-span section and hence it is equal to the moment, M'_c, at the center of the conjugate beam, which is the moment of the couple shown in Fig. 143(c). Thus,

$$\Delta=M'_c=\frac{1}{16}\frac{Pl^2}{EI}\cdot\frac{2}{3}\frac{l}{2}=\frac{1}{48}\frac{Pl^3}{EI}.$$

Further, since the conjugate beam is in equilibrium the values of R'_1 and R'_2 may be found from the equations $\Sigma F'=0$ and $\Sigma M'=0$. Thus, from $\Sigma M'=0$, or from the conditions of symmetry, we obtain, $R'_1=R'_2$. And from $\Sigma F'=0$ we obtain the values of R'_1 and R'_2, which are equal to the vertical shears at the ends of the conjugate beam and hence equal in magnitude to the slopes, θ_A and θ_B, at the ends of the " given " beam. Thus,

$$\Sigma F'=2R'_1-\frac{1}{2}\frac{Pl}{4EI}\cdot l=0,$$

whence,

$$R'_1=\frac{1}{16}\frac{Pl^2}{EI},$$

as is evident from inspection of Fig. 143(c). Therefore,

$$\theta_A = \frac{1}{16}\frac{Pl^2}{EI}.$$

Deflection at Any Point.—The deflection y of the "given" beam at a distance x from the left support is equal to the bending moment M'_x of the conjugate beam which, as indicated in Fig. 144, is

$$y = M'_x = R'_1 x - \frac{1}{2}\frac{Px^2}{2}\cdot\frac{1}{3}x$$

$$= \frac{Pl^2 x}{16EI} - \frac{1}{12}\frac{Px^3}{EI} = \frac{P}{4EI}\left(\frac{l^2 x}{4} - \frac{x^3}{3}\right).$$

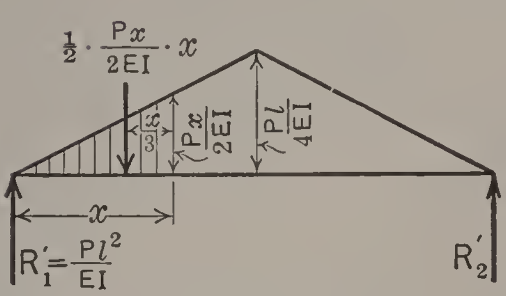


FIG. 144.

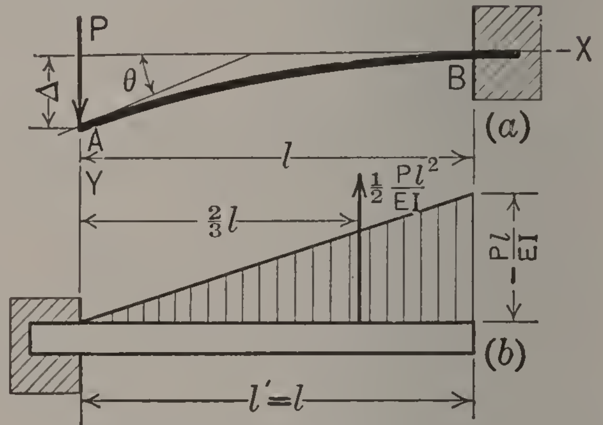


FIG. 145.—Deflection of cantilever beam, load at end; conjugate-beam method.

This is the equation of the elastic curve of the left half of the given beam.

70. Cantilever Beam; Concentrated Load at End.—The given beam is assumed to have a constant cross-section; it bends as shown in Fig. 145(a). The conjugate beam is shown in Fig. 145(b); the distributed load on the conjugate beam is the $\frac{M}{EI}$-diagram and is an upward load since M is negative. The end conditions of the conjugate beam are found by making the beam satisfy the fundamental equations of Art. 68. Thus, since $M' = y$ and $V' = \theta$, and, further, since y and θ are not zero at the free end of the "given" beam, then the moment M'_0 and the shear V'_0 at the left end of the conjugate beam are not zero. M'_0 and V'_0 may be produced by fixing the beam at the left end. On the other hand, since the θ and y at the right end of the "given" beam are zero, the M' and V' at the right end of the conjugate beam must be zero, and this condition will exist if the right end is free. Therefore, the conjugate beam corresponding to a cantilever beam is another cantilever beam with the end conditions interchanged.

Now the maximum deflection, Δ, of the "given" beam is equal to the moment M'_0 at the left end of the conjugate beam. Thus,

$$\Delta = M'_0 = \frac{1}{2}\frac{Pl^2}{EI}\cdot\frac{2}{3}l = \frac{1}{3}\frac{Pl^3}{EI}.$$

The slope, θ, at the free end of the given beam may be found, if desired, by calculating the shear V'_0 at the left end of the conjugate beam; the value found is

$$\theta = V'_0 = \frac{1}{2}\frac{Pl^2}{EI}.$$

The elastic curve equation may be found in a manner similar to that used in the preceding article.

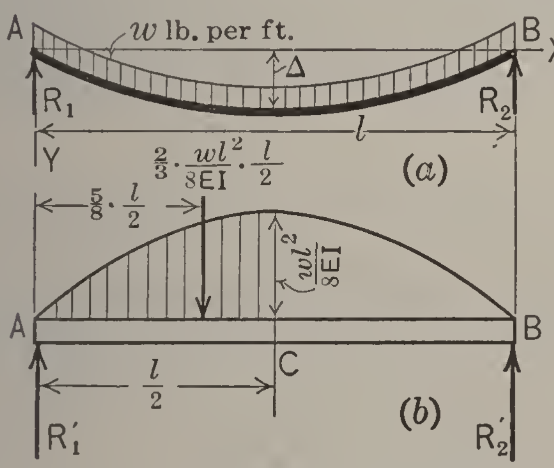


FIG. 146.—Deflection of simple beam, uniform load; conjugate-beam method.

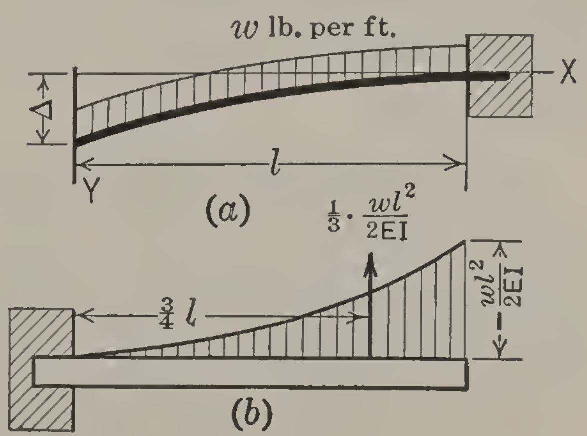


FIG. 147.—Deflection of cantilever beam, uniform load; conjugate-beam method.

71. Simple Beam; Load Distributed Uniformly.—The beam is assumed to have a constant cross-section; it deflects as shown in Fig. 146(*a*). The corresponding conjugate beam is shown in Fig. 146(*b*), the distributed load being the $\frac{M}{EI}$-diagram, which is a parabolic area, the maximum ordinate of the diagram being $\frac{1}{8}\frac{wl^2}{EI}$ (see Prob. 88).

The maximum deflection Δ is equal to the moment at the center of the conjugate beam, which is the moment of the couple shown in Fig. 146(*b*). Thus,

$$\Delta = M'c = \frac{2}{3}\cdot\frac{1}{8}\frac{Wl}{EI}\cdot\frac{l}{2}\cdot\frac{5}{8}\frac{l}{2}$$

$$= \frac{5}{384}\frac{wl^4}{EI} = \frac{5}{384}\frac{Wl^3}{EI}.$$

72. Cantilever Beam; Load Distributed Uniformly.—The beam has a constant cross-section; it deflects as shown in Fig. 147(a). The corresponding conjugate beam is shown in Fig. 147(b). The maximum deflection Δ is equal to the moment, M'_0, at the left end of the conjugate beam. Thus,

$$\Delta = M'_0 = \frac{1}{3}\cdot\frac{wl^3}{2EI}\cdot\frac{3}{4}l = \frac{1}{8}\frac{wl^4}{EI} = \frac{1}{8}\frac{Wl^3}{EI}.$$

73. Simple Beam; Concentrated Load at Any Point.—Let the load act at a distance a from the left end and b from the right end; the deflected beam is shown in Fig. 148(a). The $\frac{M}{EI}$-diagram is shown in Fig. 148(b) as a distributed load acting on the conjugate beam; the "given" beam is assumed to have a constant cross-section.

The total load is $\frac{1}{2}\frac{Pab}{EIl}\cdot l$ and its action line is at a distance of $\frac{1}{3}(l+a)$ from the left end. From the equations of equilibrium, $\Sigma F'=0$ and $\Sigma M'=0$, we find the values of R'_1 and R'_2 (and hence of θ_a and θ_b) to be,

$$R'_1 = \frac{1}{3}\frac{l+b}{l}\frac{Pab}{2EI} \quad \text{and} \quad R'_2 = \frac{1}{3}\frac{l+a}{l}\frac{Pab}{2EI}.$$

The deflection y at the distance x from the left end of the given beam is equal to the bending moment, M'_x, in the conjugate beam. Thus,

$$y = M'_x = R'_1x - \frac{1}{2}\frac{Pbx^2}{EIl}\frac{1}{3}x$$

$$= \frac{1}{3}\frac{l+b}{l}\frac{Pab}{2EI}x - \frac{1}{6}\frac{Pbx^3}{lEI}$$

$$= \frac{Pbx}{6EIl}[a(l+b) - x^2], \qquad (129)$$

which is the elastic curve equation of that portion of the beam to the left of the load P.

The maximum deflection may be found as follows: The maxmum moment in the conjugate beam is equal to the maximum deflection in the given beam; now the section of maximum moment in the conjugate beam is the section for which the vertical shear in the conjugate beam is zero. Thus, the value of x' in the following equation locates the point of maximum deflection in the given beam.

$$V'_x = R'_1 - \frac{Pbx^2}{2lEI} = 0$$

$$= \frac{1}{3}\frac{l+b}{l}\frac{Pab}{2EI} - \frac{Pbx^2}{2lEI} = 0,$$

whence,

$$x = \sqrt{\tfrac{1}{3}a(l+b)}. \qquad (130)$$

If this value of x is substituted in equation (129), the resulting value of y is the maximum deflection Δ, (see Art. 50 for the value of Δ).

PROBLEM

151. Find, from equations (129) and (130), the maximum deflection of a simple beam subjected to a concentrated load at the center of the span, and compare the result with that given in Art. 47.

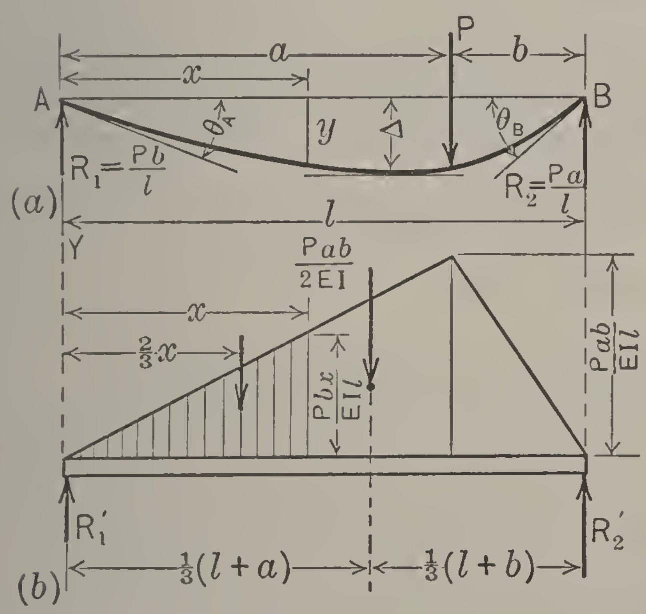


FIG. 148.—Deflection of simple beam, load at any point; conjugate-beam method.

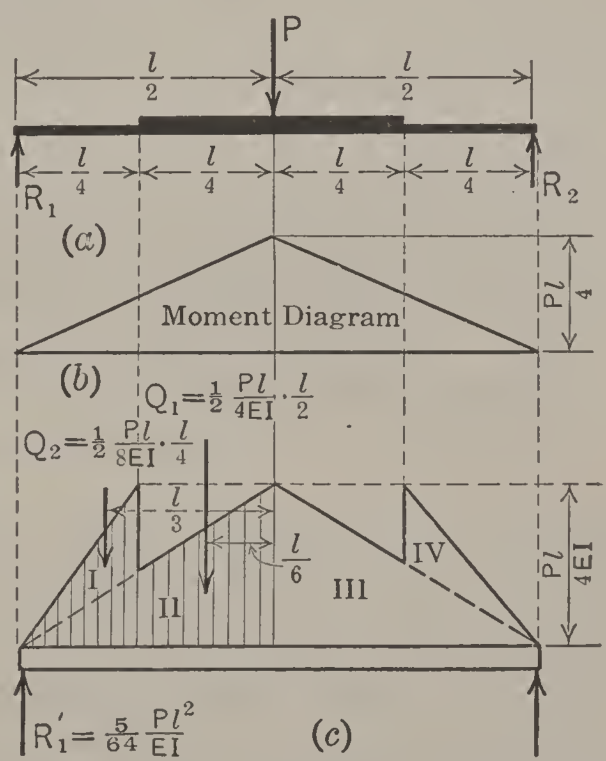


FIG. 149.—Deflection of beam with variable section; conjugate-beam method.

74. Simple Beam; Cross-section not Constant.—Let a concentrated load act at the center of a simple beam, and let the moment of inertia for any section in the center half of the beam be I and for any section in the outer quarters $\frac{1}{2}I$ as indicated in Fig. 149(a). The moment diagram is shown in Fig. 149(b), and the $\frac{M}{EI}$-diagram is shown in Fig. 149(c) acting as a load on the conjugate beam. The $\frac{M}{EI}$-diagram may be divided into four triangles, I, II, III and IV, as shown in Fig. 149(c). From the equilibrium equation $\Sigma F'=0$, we have

$$R'_1=Q_1+Q_2=\frac{5}{64}\frac{Pl^2}{EI},$$

and the maximum deflection Δ, which is equal to the bending moment $M'c$ at the center of the conjugate beam is,

$$\begin{aligned}\Delta=M'c&=R'_1\frac{l}{2}-Q_1\frac{1}{6}l-Q_2\frac{1}{3}l\\&=\left(\frac{5}{128}-\frac{1}{96}-\frac{1}{192}\right)\frac{Pl^3}{EI}\\&=\frac{3}{128}\frac{Pl^3}{EI}.\end{aligned}$$

PROBLEM

152. A simple beam is subjected to a concentrated load P at the center of the span; the moment of inertia of the left half of the beam is I and of the right half is $\frac{1}{2}I$. Derive the expression for the maximum deflection of the beam.

Note.—The problems in Chapter VI may also be used in connection with this chapter.

CHAPTER IX

STATICALLY INDETERMINATE BEAMS

(Moment-area Method)

75. Introduction.—As explained in Art. 51, a statically indeterminate beam is one for which the number of reactions is greater than the number of equations of equilibrium; in order to determine the reactions for such a beam the equation of the elastic curve of the beam is needed, in addition to the equations of equilibrium. The use, by the double integration method, of the general equation of the elastic curve for this purpose, is discussed in Chapter VII. In this chapter the moment-area method will be used to furnish sufficient equations, in addition to the equations of equilibrium, to determine the reactions of statically indeterminate beams. Further, two methods of using moment-areas will be employed; namely, the slope-deviation method and the conjugate-beam method; the equations and theorems used in the slope-deviation method are stated in Art. 63, and the equations used in the conjugate-beam method are given in Art. 68.

Slope-deviation Method

76. Beam Fixed at Both Ends; Load Concentrated at Mid-span.—The beam is shown in its deflected form in Fig. 150(*a*). The beam may be considered to have been a simple beam to which negative end-moments have been applied which cause zero slopes at the ends as indicated in Fig. 150(*b*). The moment-diagram may be found by superimposing the positive moment-diagram for a simple beam subjected to a concentrated load at the mid-span, the maximum ordinate to which is at the mid-span and equal to $\frac{Pl}{4}$, and a negative moment-diagram consisting of a rectangle, the constant ordinate to which is the unknown end-moment M_0; this superimposed diagram is shown in Fig. 150(*c*), the end-moments being known to be equal from the conditions of symmetry (or of equilibrium). Since the beam has a constant cross-section, Fig. 150(c) also represents the $\frac{M}{EI}$-diagram. Since the beam is fixed at both ends, we have

$$\text{Change of slope from } A \text{ to } B = \text{zero},$$

and hence, from Theorem I of Art. 63, we have

$$\text{Total area of } \frac{M}{EI}\text{-diagram} = 0,$$

that is,

$$\frac{1}{EI}\left(\frac{1}{2}\frac{Pl}{4}\cdot l + M_0 l\cdot\right) = 0,$$

and hence,

$$M_0 = -\tfrac{1}{8}Pl.$$

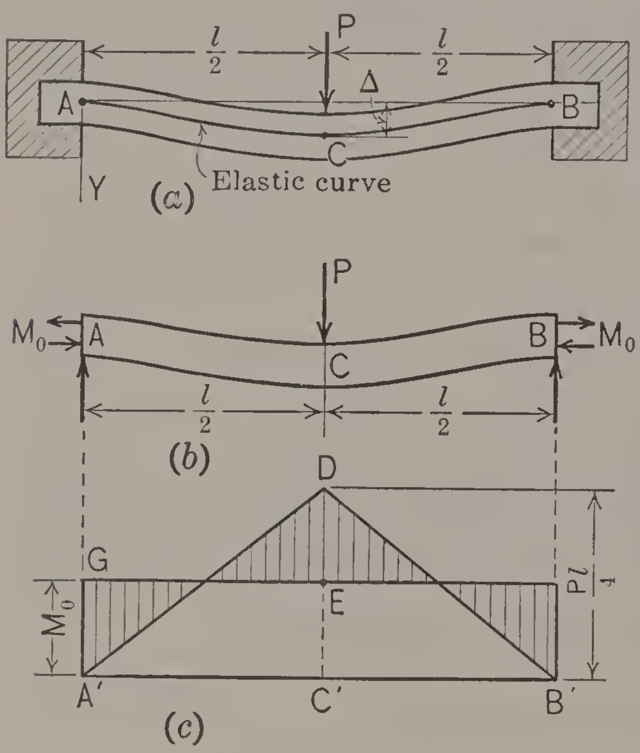


FIG. 150.—Moments at ends of fixed beam; slope-deviation method.

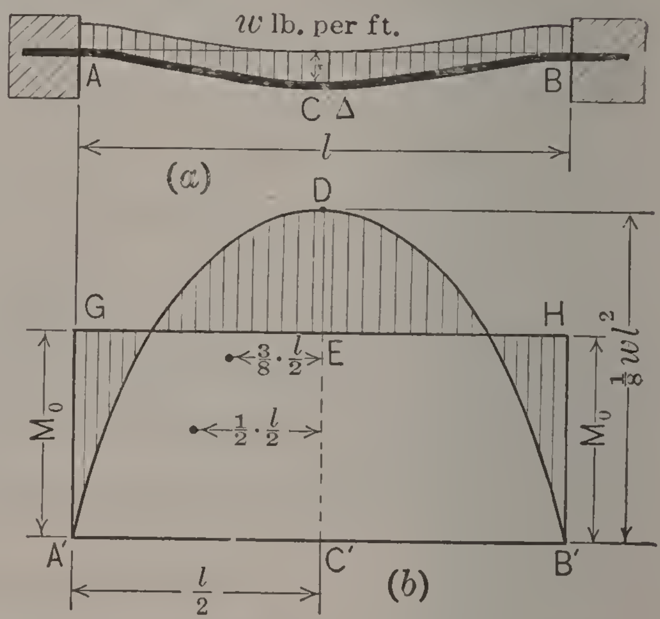


FIG. 151.—Moments at ends of fixed beam; slope-deviation method.

According to Theorem II of Art. 63 the maximum deflection, Δ, which occurs at C (Fig. 150a), is

$$\Delta = \text{moment of } \frac{M}{EI}\text{-diagram from } A' \text{ to } C' \text{ about } C'$$

$$= \frac{1}{EI}\text{ (moment of area } A'DC' \text{ about } C'D + \text{moment of area } A'GEC' \text{ about } C'E)$$

$$= \frac{1}{EI}\left(-\frac{1}{2}\frac{Pl}{4}\cdot\frac{l}{2}\cdot\frac{1}{3}\frac{l}{2} + \frac{Pl}{8}\cdot\frac{l}{2}\cdot\frac{l}{4}\right)$$

$$= \frac{Pl^3}{192EI}.$$

77. Beam Fixed at Both Ends; Load Uniformly Distributed.—The deflected beam is shown in Fig. 151(a). The $\frac{M}{EI}$-diagram is shown in Fig. 151(b); it is considered to be composed of two parts, as in the preceding article, consisting of the positive area $A'DB'$ and the negative area $A'GHB'$.

Since the change in slope from A to B (Fig. 151a) is equal to zero we have, from Theorem I of Art. 63,

$$\text{Total area of } \frac{M}{EI}\text{-diagram} = 0,$$

that is,

$$\frac{2}{3}\cdot\frac{wl^2}{8EI}\cdot l + \frac{M_0 l}{EI} = 0$$

Hence,

$$M_0 = -\frac{wl^2}{12}.$$

And from Theorem II of Art. 63, the maximum deflection, Δ, is

$$\Delta = \text{moment of area } A'DC' \text{ about } C'D - \text{moment of area } A'GEC' \text{ about } C'E$$

$$= -\frac{2}{3}\frac{wl^2}{8EI}\cdot\frac{l}{2}\cdot\frac{3}{16}l + \frac{l}{2}\left(\frac{wl^2}{12EI}\right)\cdot\frac{l}{4}$$

$$= \frac{wl^4}{384EI}.$$

78. Beam Fixed at One End, Supported at Other End; Load Concentrated at Mid-span.—The deflected beam is shown in Fig. 152(a), and the $\frac{M}{EI}$-diagram considered to be made up of two parts, as discussed in Art. 76, is shown in Fig. 152(b). Thus the moment-diagram consists of a triangle with the maximum ordinate at the center of the span, the same as that for a simple beam with a concentrated load P at the mid-span, and a negative triangular diagram the ordinates to which vary from M_0 at the wall to zero at the left end. Further, since the cross-section of

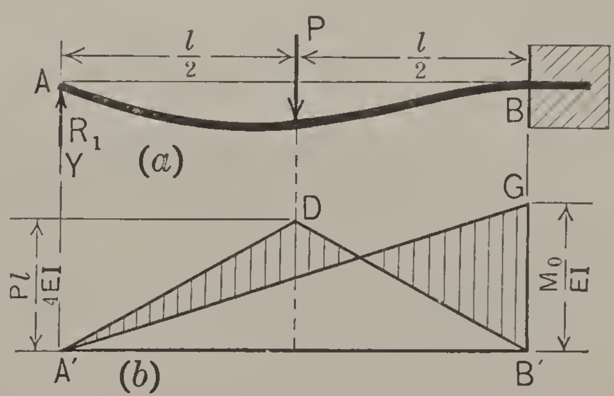


FIG. 152.—Moment at fixed end of beam; slope-deviation method.

the beam is constant, the $\frac{M}{EI}$-diagram has the same form as the moment-diagram.

Since the tangential deviation, t_A, of the point A from a tangent drawn at B is equal to zero, we have, from Theorem II of Art. 63,

$$t_A = \text{moment of } \frac{M}{EI}\text{-diagram about } A' = 0$$

$$= (\text{moment of area } A'DB' \text{ about } A' + \text{moment of area } A'GB' \text{ about } A') = 0$$

$$= \frac{1}{2}\frac{Pl}{4EI}\cdot l\cdot\frac{l}{2} + \frac{1}{2}\cdot\frac{M_0 l}{EI}\cdot\frac{2}{3}l = 0.$$

Hence,

$$M_0 = -\tfrac{3}{16}Pl.$$

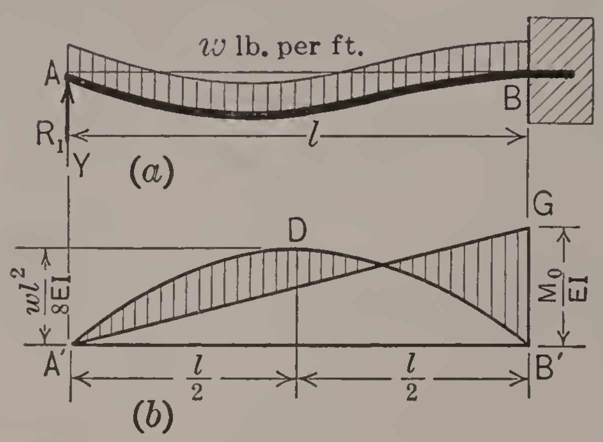


FIG. 153.—Moment at fixed end of beam; slope-deviation method.

79. Beam Fixed at One End, Supported at Other End; Load Uniformly Distributed.—The deflected beam is shown in Fig. 153(a) and the $\frac{M}{EI}$-diagram is shown in Fig. 153(b). The tangential deviation of the point A from a tangent at B is equal to zero. Hence, from Theorem II of Art. 63 we have

$$t_A = \text{moment of } \frac{M}{EI}\text{-diagram about } A' = 0$$

$$= \text{moment of area } A'DB' \text{ about } A' + \text{moment of area } A'GB' \text{ about } A' = 0$$

$$= \frac{2}{3}\frac{wl^2}{8EI}\cdot l\cdot\frac{l}{2} + \frac{1}{2}\frac{M_0}{EI}\cdot l\cdot\frac{2}{3}l = 0.$$

Hence,

$$M_0 = -\frac{wl^2}{8}.$$

80. Continuous Beam; Theorem of Three Moments.—Let Fig. 154(a) represent a continuous beam subjected to uniformly distributed loads; the beam has a constant cross-section and rests on supports that are on the same level.

The elastic curve of two adjacent spans of the beam is shown in Fig. 154(*b*). If, over the supports, hinges are introduced, as indicated in Fig. 154(*c*) and then external end-moments, M_1, M_2 and M_3, are applied as shown, the beam will act in all ways as the original beam acts, provided that the values of M_1, M_2 and M_3 are the values of the bending moments at the supports of the original beam. The moment diagram formed by superimposing the positive and negative moments, caused by the loading shown in Fig. 154(*c*) is represented in Fig. 154(*d*).

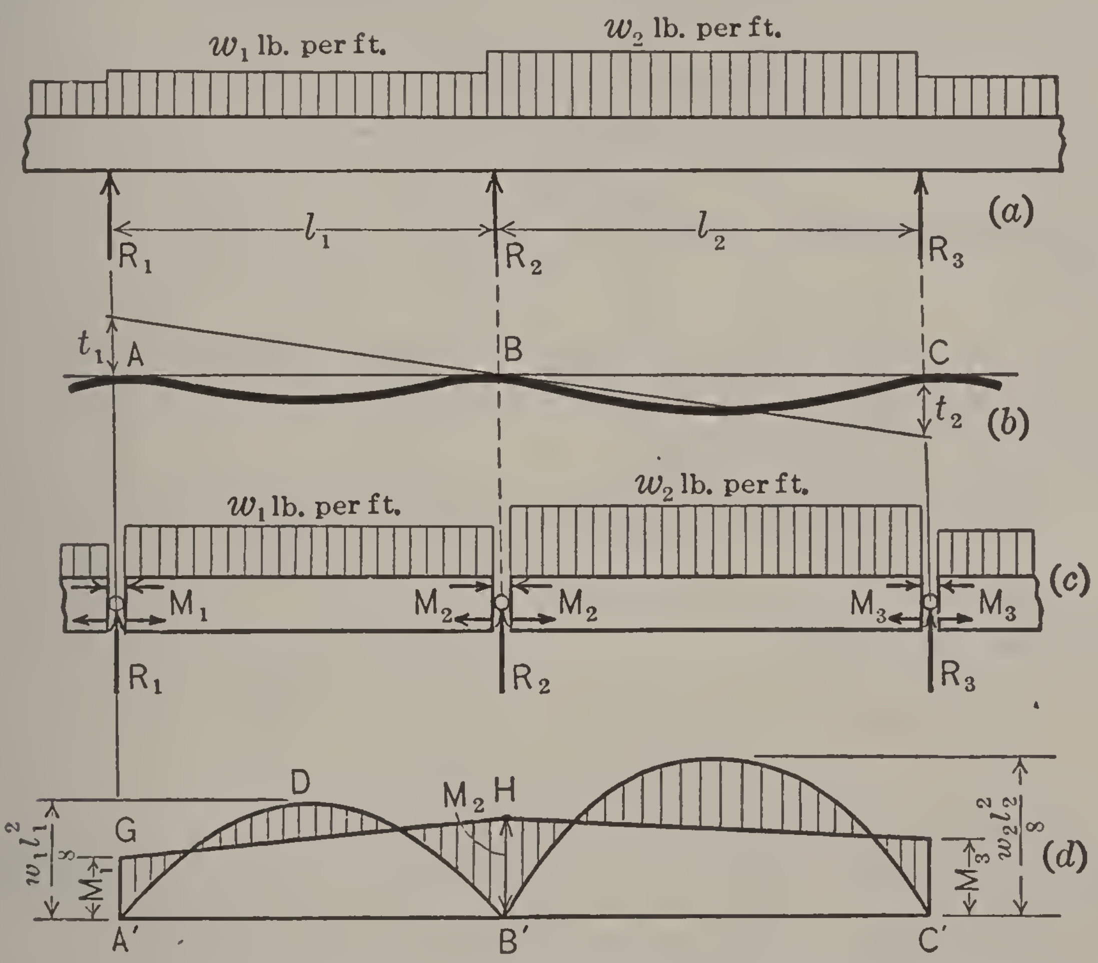


Fig. 154.—Moments at supports of continuous beam; slope-deviation method.

Let a tangent be drawn to the elastic curve at the support B (Fig. 154*b*); since the elastic curve is continuous over the support, the tangent to the elastic curve at B is common to the two curves of the adjacent spans l_1 and l_2.

Now, from Theorem II of Art. 63, the tangential deviation t_1 of the point A from a tangent at B is

$$t_1 = \frac{1}{EI} \text{ (moment, about } A'\text{, of area } A'DB' + \text{moment about } A'\text{, of area } A'GHB'\text{)}$$

$$=\frac{1}{EI}\left(\frac{2}{3}\frac{w_1l_1^2}{8}\cdot l_1\cdot\frac{l_1}{2}+\frac{1}{2}M_1l_1\frac{l_1}{3}+\frac{1}{2}M_2l_1\cdot\frac{2l_1}{3}\right)$$

$$=\frac{1}{EI}\left(\frac{w_1l_1^4}{24}+\frac{M_1l_1^2}{6}+\frac{M_2l_1^2}{3}\right).$$

Similarly, taking moments, about C', of areas of the moment diagram for the next span, we have

$$t_2=\frac{1}{EI}\left(\frac{2}{3}\frac{w_2l_2^2}{8}\cdot l_2\cdot\frac{l_2}{2}+\frac{1}{2}M_3l_2\cdot\frac{l_2}{3}+\frac{1}{2}M_2\cdot l_2\frac{2l_2}{3}\right)$$

$$=\frac{1}{EI}\left(\frac{w_2l_2^4}{24}+\frac{M_2l_2^2}{3}+\frac{M_3l_2^2}{6}\right).$$

Now, from similar triangles, $\dfrac{t_1}{l_1}=\dfrac{-t_2}{l_2}$,

whence,

$$\frac{w_1l_1^3}{24}+\frac{M_1l_1}{6}+\frac{M_2l_1}{3}=-\left(\frac{w_2l_2^3}{24}+\frac{M_3l_2}{6}+\frac{M_2l_2}{3}\right),$$

or,

$$\frac{M_1l_1}{6}+\frac{M_2(l_1+l_2)}{3}+\frac{M_3l_2}{6}=-\frac{w_1l_1^3}{24}-\frac{w_2l_2^3}{24}.$$

Therefore,

$$M_1l_1+2M_2(l_1+l_2)+M_3l_2=-\frac{w_1l_1^3}{4}-\frac{w_2l_2^3}{4}.$$

This equation expresses the theorem of three moments for a continuous beam subjected to uniform loads, the supports being on the same level. For use of the theorem see Art. 58.

Conjugate-beam Method

81. Beam Fixed at Both Ends; Load Distributed Uniformly. —The deflected beam is shown in Fig. 155(a). In Fig. 155(b) the beam is shown acted on by an equivalent force system which converts the beam into a simple beam subjected to end-moments M_0. The moment-diagram for the beam in Fig. 155(b), and hence also for the original beam, is shown in Fig. 155(c); the positive moment diagram of the vertical forces is a parabola the same as that for a simple beam, and the negative moment diagram of the end-moments is a rectangle. Further, since the beam has a constant cross-section, the $\frac{M}{EI}$-diagram has the same form as the moment-diagram.

Now, as discussed in Art. 70, the conjugate beam for a fixed-ended "given" beam is a beam with free ends, that is, the $\frac{M}{EI}$-diagram, acting alone as a distributed load on the conjugate beam, must hold the conjugate beam in equilibrium as indicated in Fig. 155(*d*), and hence the $\frac{M}{EI}$-load diagram must be so adjusted that the sum of all the loads is equal to zero ($\Sigma F'=0$, Art. 68); that is, in Fig. 155(*c*), the $\frac{M}{EI}$-load of the parabolic diagram must be equal and opposite to the $\frac{M}{EI}$-load of the rectangular diagram. Hence

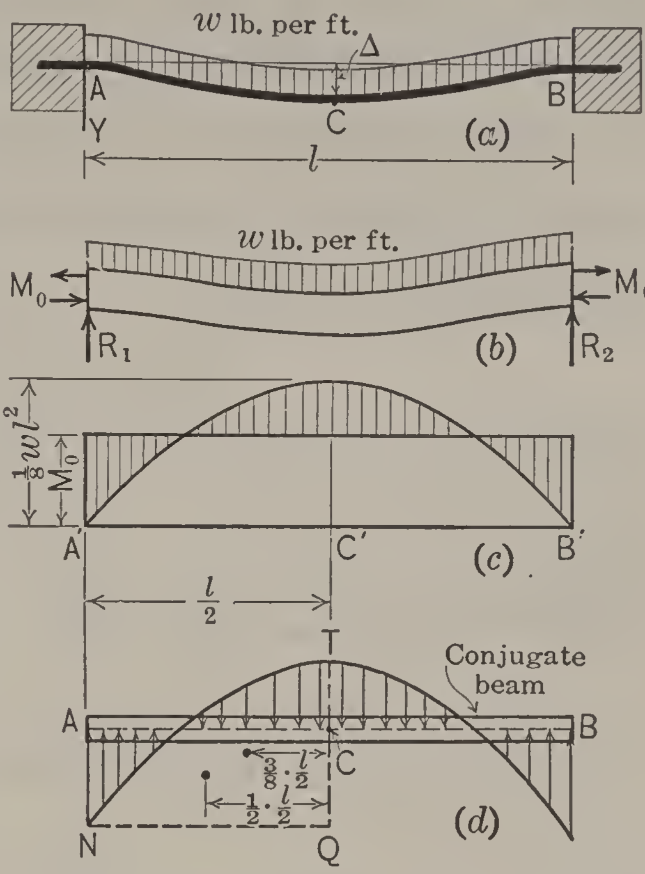


FIG. 155.—Moments at ends of fixed beam; conjugate-beam method.

$$\frac{2}{3}\frac{wl^3}{8EI}\cdot l=-\frac{M_0}{EI}l,$$

whence,

$$M_0=-\tfrac{1}{12}wl^2.$$

Maximum Deflection.—The maximum deflection Δ of the given beam is equal to the bending moment M'_c at the center of the conjugate beam. In finding M'_c the load to the left of the center may be considered to consist of two parts: the rectangle $ACQN$ (Fig. 155*d*) acting as an upward load, and the parabolic area NTQ acting as a downward load; these two areas are equal in magnitude since the beam is in equilibrium. Taking moments about C' we have,

$$\Delta=M'_c=\frac{1}{12}\frac{wl^2}{EI}\cdot\frac{l}{2}\cdot\frac{l}{4}-\frac{2}{3}\cdot\frac{1}{8}\frac{wl^2}{EI}\cdot\frac{l}{2}\cdot\frac{3}{8}\frac{l}{2}$$

$$=\frac{wl^4}{384EI}=\frac{Wl^3}{384EI}.$$

82. Beam Fixed at Both Ends; Load Concentrated at Mid-span.—The deflected beam is shown in Fig. 156(a). A force system that would produce the same deflections as occur in the "given" beam is shown in Fig. 156(b), and the superimposed moment-diagram is shown in Fig. 156(c). The conjugate beam (Fig. 156d) for a fixed-ended "given" beam has free ends (see Art. 70), and hence the $\frac{M}{EI}$-diagram must hold the conjugate beam in equilibrium; that is, the total $\frac{M}{EI}$-load must equal zero ($\Sigma F'=0$, Art. 68). Hence,

$$\frac{M_0}{EI}l+\frac{1}{2}\cdot\frac{1}{4}\frac{Pl}{EI}\cdot l=0,$$

whence,

$$M_0=-\tfrac{1}{8}Pl.$$

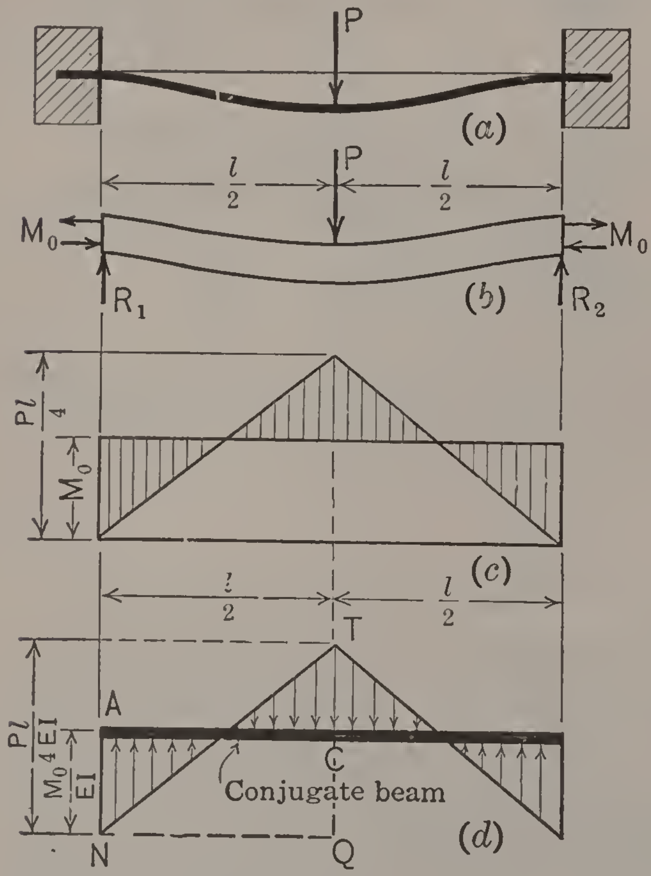


Fig. 156.—Moments at ends of fixed beam; conjugate-beam method.

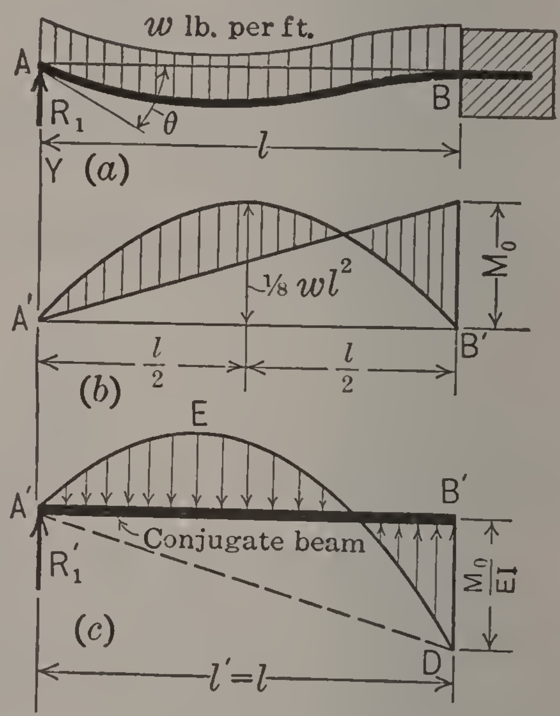


Fig. 157.—Moment at fixed end of beam; conjugate-beam method.

The maximum deflection Δ is equal to the bending moment M'_c at the center of the conjugate beam. Thus,

$$\Delta=M'_c=(\text{moment } ACQN-\text{moment } NTQ)$$

$$=\frac{1}{8}\frac{Pl}{EI}\frac{l}{2}\cdot\frac{l}{4}-\frac{1}{2}\frac{1}{4}\frac{Pl}{EI}\cdot\frac{l}{2}\cdot\frac{1}{3}\frac{l}{2}$$

$$=\frac{1}{192}\frac{Pl^3}{EI}.$$

83. Beam Fixed at One End; Supported at Other End; Load Uniformly Distributed.—The deflected beam is shown in Fig. 157(*a*). The superimposed moment-diagram is shown in Fig. 157(*b*).

Since the deflection and slope at the right end of the "given" beam are zero, the shear and moment at the right end of the conjugate beam must be zero, and hence the end is free (see Art. 70); whereas, the left end of the conjugate beam has zero moment ($M'=y=0$) but not zero shear ($V'=\theta$), and hence is supported at the left end, and is subjected to the distributed $\frac{M}{EI}$-load as indicated in Fig. 157(*c*).

The value of M_0 may be found by applying one of the equations of equilibrium ($\Sigma M'=0$). Thus if moments are taken about the left end of the conjugate beam the moment of the triangular area $A'B'D$ plus the moment of the parabolic area $A'ED$ must equal zero. Hence,

$$\frac{1}{2}\frac{M_0}{EI}l\cdot\frac{2}{3}l+\frac{2}{3}\cdot\frac{1}{8}\frac{wl^2}{EI}l\cdot\frac{l}{2}=0,$$

whence,

$$M_0=-\frac{1}{8}wl^2.$$

Now R'_1 (and hence θ) may be found, if desired, from the other equilibrium equation ($\Sigma F'=0$).

84. Continuous Beam; Theorem of Three Moments.—Fig. 158(*a*) shows two spans of a continuous beam subjected to distributed loads; the supports are on the same level, and the beam has a constant cross-section. If, over the supports, hinges are introduced and then end-moments, M_1, M_2 and M_3, are applied, as in Fig. 158(*c*), the beam will act in all ways as the original beam acts, provided that the value of M_1, M_2, and M_3 are the values of the bending moments at the supports of the original beam.

The superimposed moment diagram for the beam in Fig. 158(*c*) (and hence also of the original beam) is shown in Fig. 158(*d*), and the $\frac{M}{EI}$-diagram will have the same form as that of the moment-diagram. The $\frac{M}{EI}$-diagram is the distributed load on the conjugate beam (Fig. 158*e*); now the conjugate beam has hinges and no supports at the points corresponding to the sections over the support of the original beam. This may be explained as follows: The deflection at the supports of the "given" beam, and therefore the moment at the supports in the conjugate beam, are zero; the slope of the given beam at a support may be different from zero (as indicated in Fig. 158*b*) but must have the same

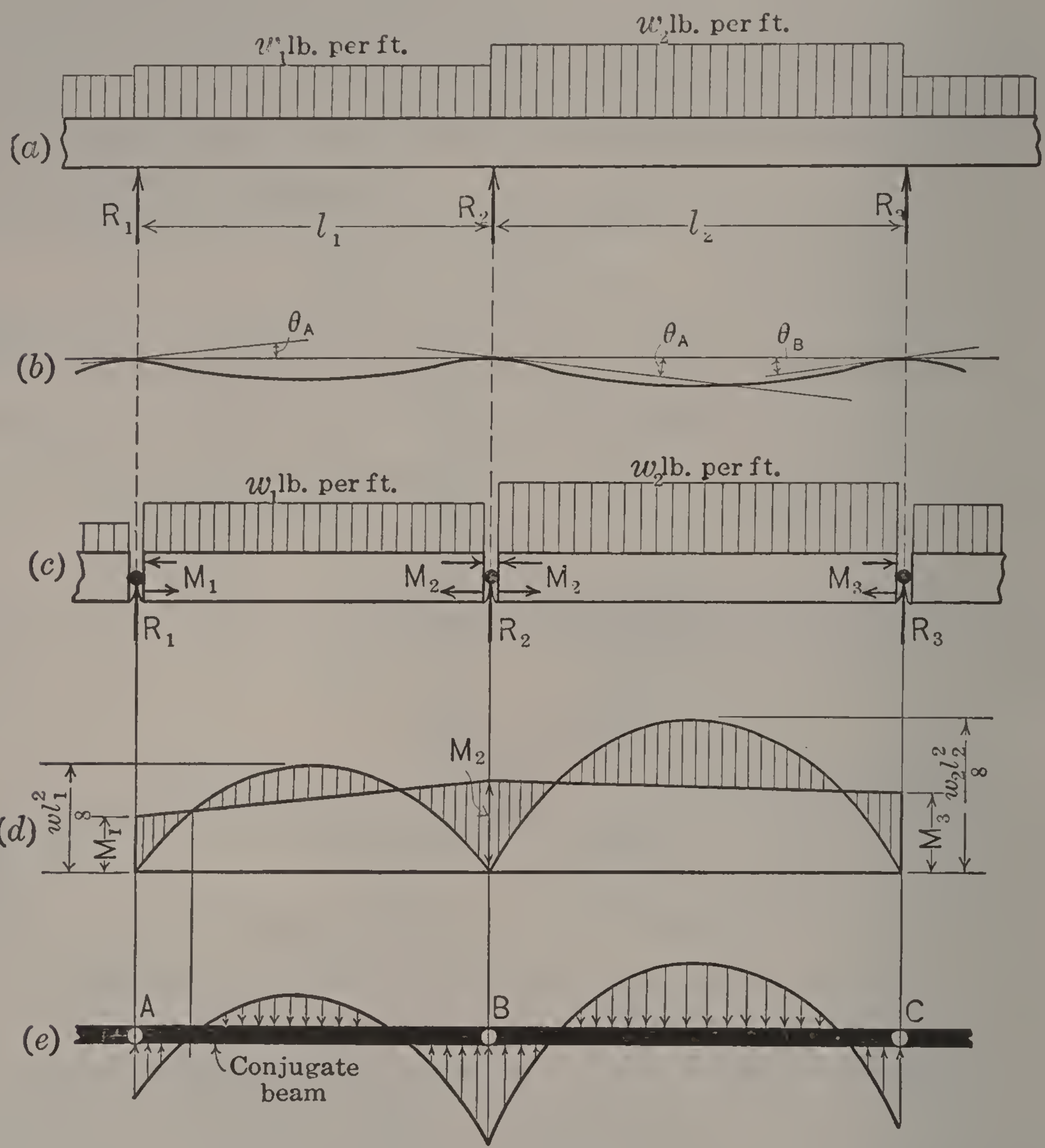


FIG. 158.—Moments at supports of continuous beam; conjugate-beam method.

value immediately to the left and to the right of the point since the beam is continuous over the support, and therefore the shear in the conjugate beam may have a value other than zero but must have the same value immediately to the left and the right of the point. An unsupported hinge is the simplest arrangement by which this condition can be established.

By expressing the fact that the $\frac{M}{EI}$-load must be so adjusted that the reaction of the support (B say, Fig. 158e) is zero, the theorem of three moments is found as follows: Let it be assumed first that all the $\frac{M}{EI}$-loads are acting downward, the reaction at B then would be such that its moment about A would be equal to the moment, about A, of the $\frac{M}{EI}$-loads on span l_1, and its moment about C would equal the moment about C of the $\frac{M}{EI}$-loads on l_2. Therefore, writing the expression for the reaction and equating it to zero, we have

$$\frac{1}{l_1}\left(\frac{2}{3}\cdot\frac{w_1 l_1^2}{8}\cdot l_1\cdot\frac{l_1}{2}+\frac{1}{2}M_2 l_1\cdot\frac{2}{3}l_1+\frac{1}{2}M_1 l_1\cdot\frac{1}{3}l_1\right)$$
$$+\frac{1}{l_2}\left(\frac{2}{3}\,\frac{w_2 l_2^2}{8}\cdot l_2\cdot\frac{l_2}{2}+\frac{1}{2}M_2 l_2\cdot\frac{2}{3}l_2+\frac{1}{2}M_3 l_2\frac{1}{3}l_2\right)=0,$$

whence

$$M_1 l_1+2M_2(l_1+l_2)+M_3 l_2=-\tfrac{1}{4}w_1 l_1^3-\tfrac{1}{4}w_2 l_2^3,$$

which expresses the theorem of three moments. For use of the theorem see Art. 58.

NOTE.—The problems in Chapter VII may also be used in connection with this chapter.

CHAPTER X

COMBINED AXIAL AND BENDING LOADS. ECCENTRIC LOADS

85. Introduction.—In the preceding chapters stresses caused by axial, torsional, and bending loads were found when the loads acted singly. Members of many structures and machines, however, are subjected to loads of two or more of these types (or the actual loads may conveniently be resolved into loads of two or more of these types), and the unit-stress developed at any point in the member may frequently be found by assuming that the loads act independently and hence each load is assumed to produce the same unit-stress that it would produce if it were the only load acting on the member; these unit-stresses may then be combined, if the stresses are within the proportional limit, to obtain the actual unit-stress. This principle, called the principle of superposition, will be used in this chapter in determining the normal (tensile or compressive) unit-stress at a point in a member when the member is subjected to combined axial (tensile or compressive) and bending loads, and also in determining the shearing unit-stress when the member is subjected simultaneously to a central shearing load and a torsional load.

86. A Beam Subjected to an Axial End Load.—In Fig. 159(a) is represented a simple beam of length l feet subjected to a uniformly distributed transverse load of w pounds per foot and a compressive axial load of Q pounds at its ends. Let it be required to find the maximum normal (tensile or compressive) unit-stress on a normal cross-section of the beam. The beam will be assumed to be short so that the deflection of the beam may be neglected without introducing serious error in the results, and hence the load Q may be assumed to be an axial load for each cross-section of the beam. Further, the weight of the beam is assumed to be negligible in comparison with the other loads on the beam.

The maximum axial unit-stress will occur at the mid-section since the unit-stress due to the transverse bending load is maximum at the mid-section and the direct stress due to Q is the same for all sections. Now as indicated by the free body diagram of the left half of the beam in Fig. 159(b), if Q were the only force acting, the stress would be uniformly distributed on the cross-sectional area a, the unit-stress at any point in the area being s_1. The total stress would be as_1 and would be equal and opposite to Q since it would hold Q in equilibrium. Hence,

$$Q = as_1 \quad \text{or} \quad s_1 = \frac{Q}{a}.$$

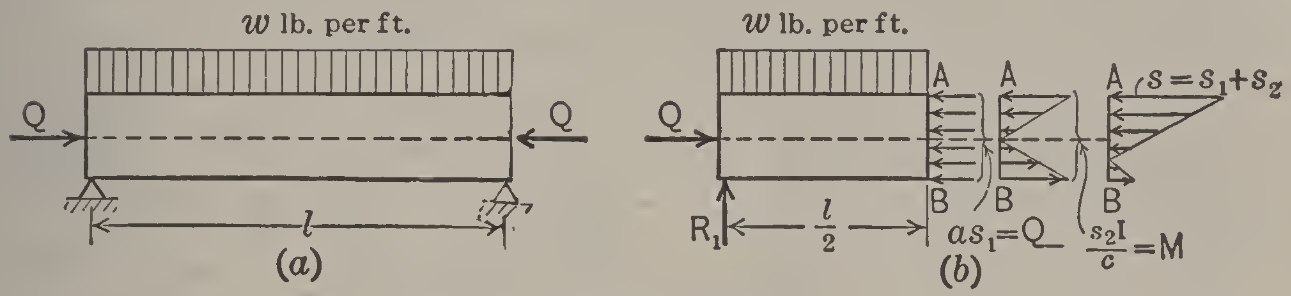


FIG. 159.—Stress in beam subjected to end-loads; deflection neglected.

If, on the other hand, the bending loads were the only forces acting, the bending moment M at the mid-section (equal to $\frac{1}{8}wl^2$, Fig. 92) would develop the resisting moment $\frac{s_2 I}{c}$ as shown in Fig. 159(b), the fiber unit-stress s_2 varying directly as the distance of the fiber from the neutral axis provided that its value does not exceed the proportional limit of the material.[1] And since the resisting moment holds the bending moment in equilibrium, the two moments are numerically equal. Hence,

$$M = \frac{s_2 I}{c} \quad \text{or} \quad s_2 = \frac{Mc}{I}.$$

The normal unit-stress at any point of the mid-section then, according to the principle of superposition, is the algebraic sum of the unit-stresses caused by the loads acting separately; the maximum unit-stress is a compressive stress and occurs on the top fiber; its value is

$$s = s_1 + s_2,$$
$$= \frac{Q}{a} + \frac{Mc}{I}.$$

[1] For other limitations and assumptions see Art. 43.

The unit-stress on the bottom fiber may be either tensile or compressive according as s_2 is larger or smaller than s_1. As shown in Fig. 159(b), s_2 is larger than s_1, therefore, the unit-stress on the bottom fiber is a tensile stress, and the surface of zero fiber-stress is some distance below the centroidal plane of the beam. Further, if the load Q were a tensile load the maximum unit-stress would occur on the bottom fiber and would be a tensile stress.

Deflection of Beam Not Negligible.—If the deflection of the beam is not negligible, the load Q (Fig. 160a) cannot be considered to be an axial load with respect to the mid-section (or any other cross-section except the end sections, see Art. 3 and Fig. 2). Let Δ denote the deflection of the axis of the beam at the section on which the stress is to be found (in this problem the mid-section).

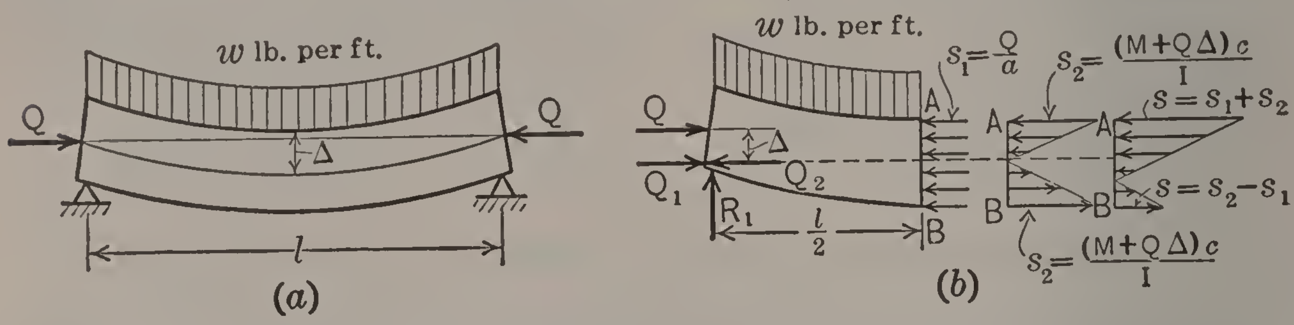


FIG. 160.—Stress in beam subjected to end loads; deflection considered.

Now if two equal opposite and collinear forces Q_1, Q_2 (each equal to Q), are applied to the beam as shown in Fig. 160(b), the force Q will be resolved into a force Q_1, which is axial with respect to the mid-section, and a couple having a moment $Q\Delta$; these two forces do not change the stresses developed on the mid-section, but simply modifies the force system so that the equations ($P=as$ and $M=\dfrac{sI}{c}$) already developed in the preceding chapters can be made to apply to this problem. Thus, since Q_1 passes through the centroid of the mid-section, the unit-stress s_1 due to Q_1 alone is constant over the section and is obtained from the equation

$$Q_1=Q=as_1,$$

as indicated in Fig. 160(b). The other forces constitute a bending moment composed of the cross-bending moment M, as in the previous problem, and the moment $Q\Delta$. The total bending moment is therefore, $M+Q\Delta$ and is held in equilibrium by the resisting moment $\dfrac{s_2I}{c}$. Therefore,

$$M+Q\Delta=\frac{s_2I}{c},$$

as indicated in Fig. 160 (b). Thus the maximum unit-stress s is,

$$s = s_1 + s_2,$$

$$= \frac{Q}{a} + (M + Q\Delta)\frac{c}{I}. \qquad \ldots \ldots \quad (131)$$

But since the value of Δ depends on the value of the total bending moment $(M+Q\Delta)$ and the value of the bending moment depends in turn on the value of Δ, a method of approximation in solving equation (131) is, as a rule, the most convenient method. Thus, the value of Δ that would be caused by the cross-bending moment, M, alone is found first (equal in this problem to $\frac{5}{384}\frac{wl^4}{EI}$, Art. 46); this value of Δ is then used in the expression $M+Q\Delta$ and this new value of the total ending moment is used to find a closer approximation to the value of Δ. This operation may be repeated as many times as desired.

ILLUSTRATIVE PROBLEM

Problem 153.—In Fig. 161(a) is shown a wall bracket, the horizontal boom BC being a 7-in. 15-lb. I-beam. The beam is pin-connected to the post

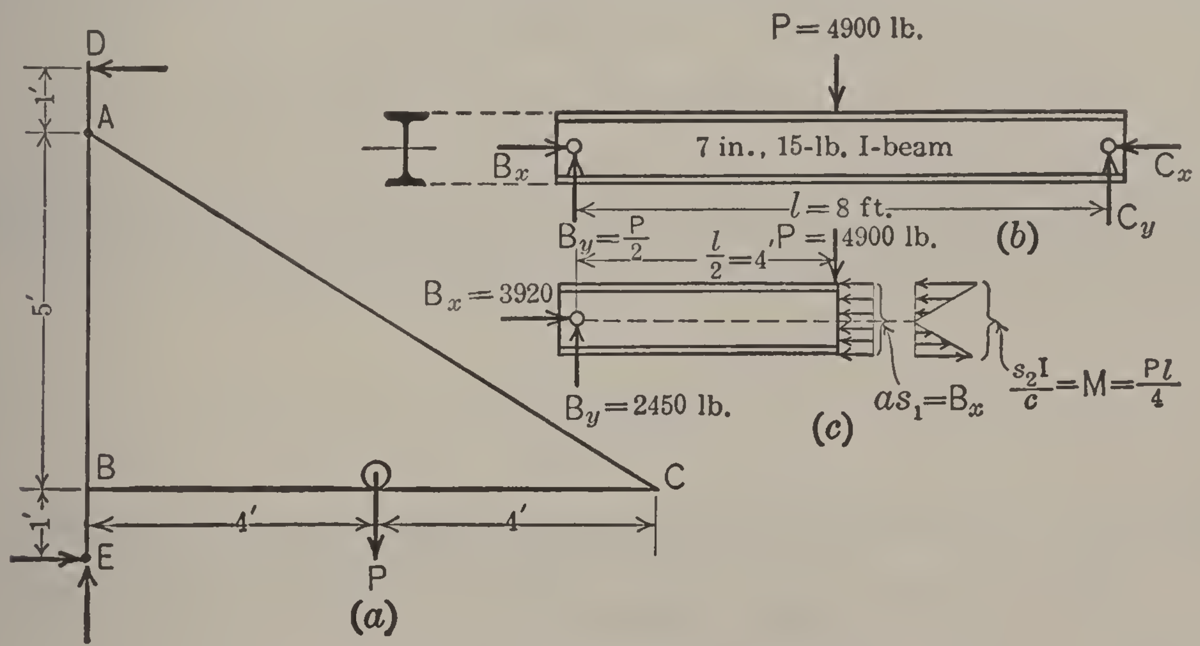


FIG. 161.—Stress in boom of wall crane.

DE at B and to the rod AC at C. If the load P is 4900 lb. find the maximum fiber unit-stress in the I-beam, assuming that the weight of the beam is negligible.

Solution.—Deflection Neglected.—The forces acting on the I-beam are shown in Fig. 161(b). The normal stresses at the mid-section due to the end load B_x and the cross-bending loads are shown in Fig. 161(c). The value of B_x and of C_x is equal to the horizontal component (x-component) of the tension in the tie rod AC, and this tension may be found by applying the conditions of equilibrium either algebraically or graphically to the forces acting on the pin at C or on the post DE; it is assumed that the student is familiar with methods of finding the forces acting on the members of pin-connected structures. The value of the tension in the rod is found to be 4600 lb. and its x-component is 3920 lb. Therefore B_x equals 3920 lb. Further, from a steel handbook the area a of the cross-section of the I-beam is found to be 4.42 sq. in. and the section modulus I/c is equal to 10.4 in.?

As indicated in Fig. 161(c) the compressive unit-stress, s, at the top of the I-beam is

$$
\begin{aligned}
s &= s_1 + s_2 \\
&= \frac{B_x}{a} + \frac{Pl}{4} \cdot \frac{c}{I} \\
&= \frac{3920}{4.42} + \frac{4900 \times 8 \times 12}{4} \times \frac{1}{10.4} \\
&= 888 + 11{,}300 \\
&= 12{,}200 \text{ lb. per sq. in.}
\end{aligned}
$$

Deflection Not Neglected.—The deflection Δ caused by the transverse forces alone is (Art. 47)

$$
\begin{aligned}
\Delta &= \frac{1}{48} \frac{Pl^3}{EI} = \frac{1}{48} \frac{4900(8 \cdot 12)^3}{30{,}000{,}000 \cdot 36.4} \\
&= 0.0825 \text{ in.}
\end{aligned}
$$

When the beam deflects, the load B_x has a moment $B_x\Delta$ with respect to the mid-section and, as explained above, the bending moment is $M + B_x\Delta$. Thus the compressive unit-stress on the top fiber of the beam is

$$
\begin{aligned}
s &= s_1 + s_2, \\
&= \frac{B_x}{a} + \left(\frac{Pl}{4} + B_x\Delta\right)\frac{c}{I} \\
&= 888 + 11{,}300 + 4900 \cdot 0.0825 \cdot \frac{1}{10.4} \\
&= 888 + 11{,}300 + 31.2.
\end{aligned}
$$

Thus, very little error is introduced in this problem by neglecting the deflection of the beam.

87. Eccentric Longitudinal Load in Plane of Symmetry.—

In Fig. 162(a) is represented a short [2] compression member acted

[2] The effect of an eccentric load on a *long* compression member (column) in which the deflection of the member must be considered, is discussed in Art. 104.

on by an eccentric load P in a plane containing an axis of symmetry [3] of each cross-section of the member, the amount of the eccentricity being denoted by e. Let it be required to find the maximum unit-stress developed on any normal cross-section, a, of the member.

The force P may be resolved into an axial load P_1 (equal to P) and a couple having a moment Pe (or P_2e since $P_1 = P_2 = P$) without affecting the stresses developed on the section (Fig. 162*b*).

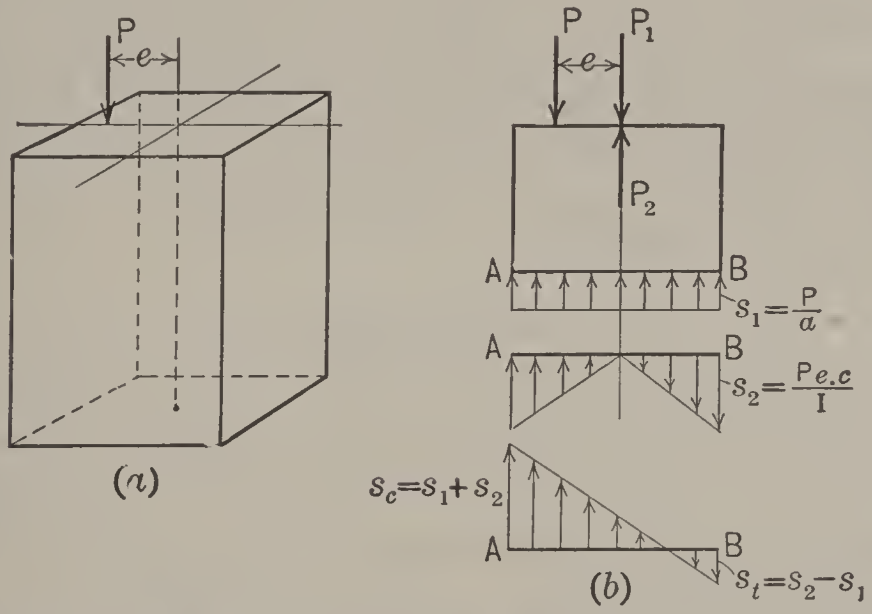


FIG. 162.—Stress due to eccentric load in plane of symmetry.

Now if P_1 were acting alone it would cause a constant unit-stress s_1 on the area a such that,

$$P_1 = P = as_1.$$

And, if the forces P and P_2 were acting alone their moment Pe would cause the resisting moment $\frac{s_2 I}{c}$ (Fig. 161*b*) such that

$$Pe = \frac{s_2 I}{c}.$$

[3] The stress caused by an eccentric load that does not act at a point on an axis of symmetry or principal axis, is discussed in Art. 88. All axes, however, are principal axes for a member having a square or circular cross-section; that is, the moments of inertia with respect to all axes in the plane of the area passing through the centroid of a square or circular area are equal.

The combined effect of the axial force and couple is to produce a maximum compressive unit-stress s such that

$$s=s_1+s_2$$

$$=\frac{P}{a}+\frac{Mc}{I}$$

$$=\frac{P}{a}+\frac{Pec}{I}$$

and the minimum stress is $\frac{P}{a}-\frac{Pec}{I}$. Or, since I may be expressed by the equation $I=ak^2$ in which k is the radius of gyration of the area a with respect to the centroidal-axis, the unit-stress at any point whose distance from the centroidal axis is c may be found from the equation

$$s=\frac{P}{a}\left(1\pm\frac{ec}{k^2}\right),$$

in which s must not exceed the proportional limit of the material.

Limitation on Eccentricity to Prevent Tensile Stress.—If the unit-stress s_2 at B (Fig. 162*b*) due to the moment Pe is greater than the unit-stress s_1 due to the axial load P_1, the resulting normal unit-stress at B will be a tensile stress. If, then, a tensile stress is to be avoided, as is usually desired in brittle material such as concrete, brick, cast iron, etc., which are relatively weak in tension, the value of e should not be greater than that found by equating s_2 equal to s_1. Thus,

$$\frac{Pec}{I}=\frac{P}{a},$$

or

$$\frac{Pec}{ak^2}=\frac{P}{a},$$

Hence

$$e=\frac{k^2}{c}.$$

Therefore, a short compression member will not be subjected to a tensile stress if the eccentric load acts on an axis of symmetry of the cross-section at a distance not greater than $\frac{k^2}{c}$ from the central axis of the member.

If the member has a rectangular cross-section the value $\frac{k^2}{c}$ is $\frac{1}{6}b$ or $\frac{1}{6}h$ where b and h are the dimensions of the cross-section. This fact is expressed by the common rule that in the design of masonry structures the load should not lie outside the middle-third of the central axes of a rectangular cross-section. If the cross-sectional area of the member is circular the value of $\frac{k^2}{c}$ is $\frac{1}{8}d$, where d is the diameter, and hence the load should not lie outside the middle-fourth of any diameter of the section if tensile stress in the member is to be avoided.

(In each of the following problems the load lies in a plane of symmetry of the cross-section of the area on which the stress is to be found.)

ILLUSTRATIVE PROBLEM

Problem 154.—The machine member shown in Fig. 163(*a*) is acted on by a force P of 3000 lb. Find the maximum normal unit-stress on the area of the section AB at the wall. The dimensions of the cross-section of the bar at section AB is $\frac{3}{4}$ in. by 3 in.

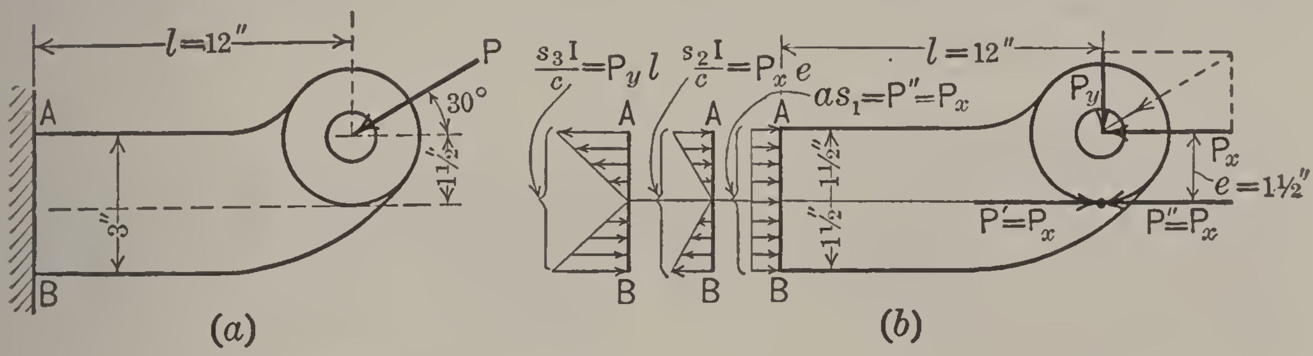


FIG. 163.

Solution. The load P may be resolved in two components, a cross-bending load P_y and a longitudinal eccentric load P_x (Fig. 163*b*); and P_x may be resolved further into an axial load P'' and a couple having a moment $P_x e$, by introducing the two equal opposite and collinear forces P' and P'', each equal to P_x, as indicated in Fig. 163(*b*).

Now since P'' is an axial load with respect to the section AB, it would, if acting alone, cause, and be held in equilibrium by, a total stress as_1, the unit-stress s_1 being constant over the area a of the section as indicated in Fig. 163(*b*). And the couple $P_x e$, if acting alone, would develop a resisting moment $\frac{s_2 I}{c}$ equal to the moment $(P_x e)$ of the couple as shown in Fig. 163(*b*). Again the cross-bending load P_y would develop a resisting moment $\frac{s_3 I}{c}$ equal to the moment $(P_y l)$. The shearing stress on the area is neglected in this discussion.

The normal unit-stress on the top fiber at A, then, is

$$
\begin{aligned}
s_A &= -s_1 - s_2 + s_3 \\
&= -\frac{P\cos 30^\circ}{2.25} - \frac{P\cos 30^\circ \cdot 1.5 \cdot 1.5}{\frac{1}{12}\,\frac{3}{4}\cdot(3)^3} + \frac{P\sin 30^\circ \cdot 12 \cdot 1.5}{\frac{1}{12}\,\frac{3}{4}\cdot(3)^3} \\
&= -\frac{2600}{2.25} - \frac{2600 \cdot 2.25}{1.69} + \frac{1500 \cdot 18}{1.69} \\
&= -1150 - 3460 + 16{,}000 \\
&= 11{,}390 \text{ lb. per sq. in., tensile stress.}
\end{aligned}
$$

And at B the unit-stress is,

$$
\begin{aligned}
s_B &= -s_1 + s_2 - s_3 \\
&= -1150 + 3460 - 16{,}000 \\
&= -13{,}690 \text{ lb. per sq. in., compressive stress.}
\end{aligned}
$$

PROBLEMS

155. In Fig. 164 is shown a machine member having a rectangular cross-section 1 in. by 4 in. It is acted on by a force P_2 of 18,000 lb. and a force P_1, the action lines of which are shown in the figure. Find the value of P_1 if the maximum tensile unit-stress is 20,000 lb. per sq. in. *Ans.* $P_1 = 8270$ lb.

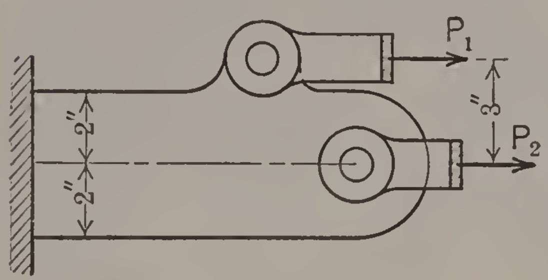


FIG. 164.

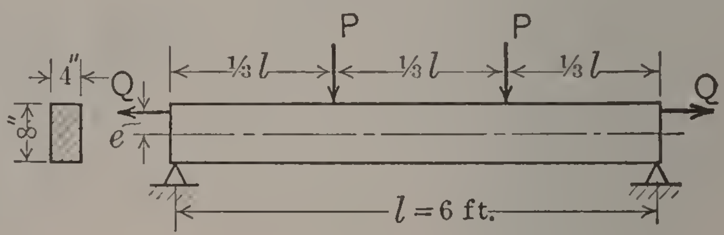


FIG. 165.

156. What value should e, in Fig. 165 have in order that the unit-stress at the top fiber within the middle third of the beam, due to the load Q shall be equal (and opposite) to the unit-stress due to the cross-bending loads; $Q = 9P$.

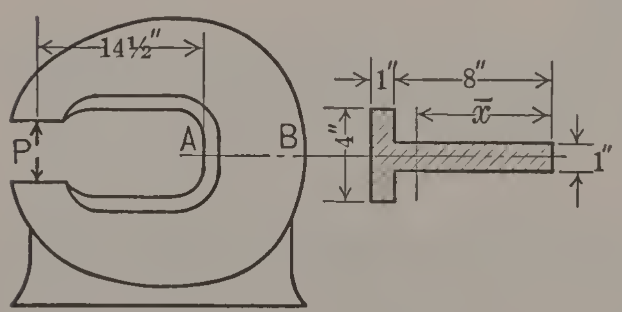


FIG. 166.

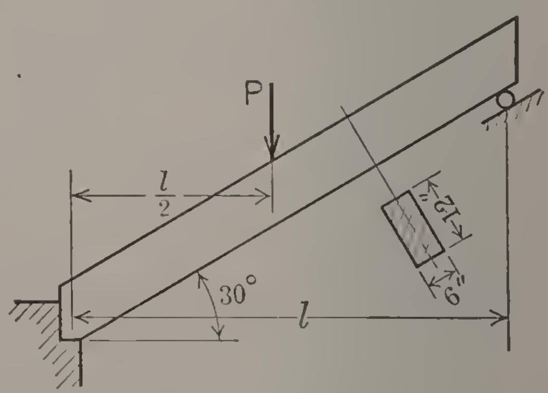


FIG. 167.

157. The frame shown in Fig. 166 is used for small riveting, punching and stamping machines. Find the unit-stress developed at A and at B when the load P is 2000 lb.

Ans. $s_A = 1470$ lb. per sq. in., tension; $s_B = 1870$ lb. per sq. in., compression.

158. The inclined beam shown in Fig. 167 carries a load P of 5000 lb. at the mid-span. If the length, l, is 10 ft., what is the maximum compressive unit-stress developed in the beam?

Ans. $s = 1080$ lb. per sq. in.

159. The post DE of the wall bracket shown in Fig. 161(a) is made of two 6-in. 15.5-lb. channel sections latticed together, the 6-in. dimensions being parallel to the direction of the boom BC. Find the maximum compressive unit-stress on a section of the post just beneath the section passing through A. Assume the load to be at the outer end of the boom BC.

160. A cast-iron machine frame shown in Fig. 168 is subjected to a load P of 8000 lb. The area of the cross-section at AB is 40 sq. in. and the centroidal axis YY is 6 in. from the outer edge of the section. The moment of inertia of the area with respect to the centroidal axis is 400 in⁴. Find the maximum tensile and compressive unit-stresses on the section AB.

Ans. $s_t = 1960$ lb. per sq. in., tension; $s_c = 2440$ lb. per sq. in., compression.

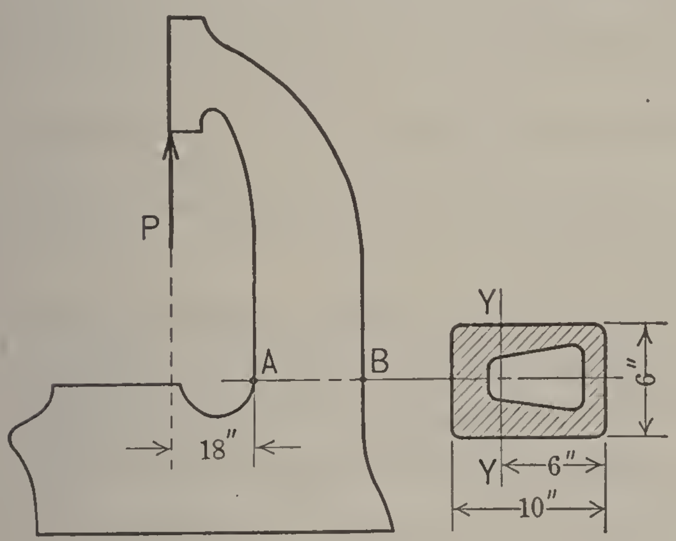


Fig. 168.

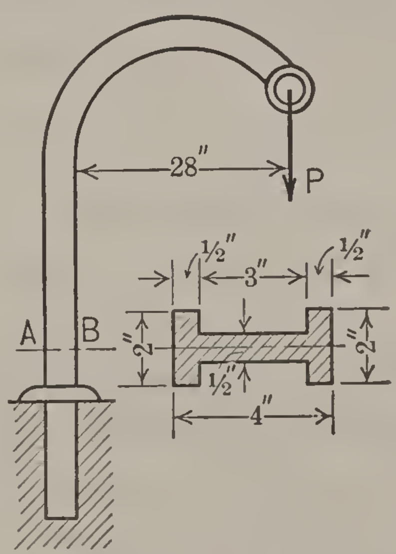


Fig. 169.

161. The small crane shown in Fig. 169 has a clear swing of 28 in. Find the load P which will cause a maximum compressive unit-stress of 9000 lb. per sq. in. on the inner edge at B. *Ans.* $P = 1090$ lb.

162. The open link shown in Fig. 170 is made of a steel bar having a diameter of 2 in. If the tensile unit-stress at A is 18,000 lb. per sq. in. what is the value of the load P?

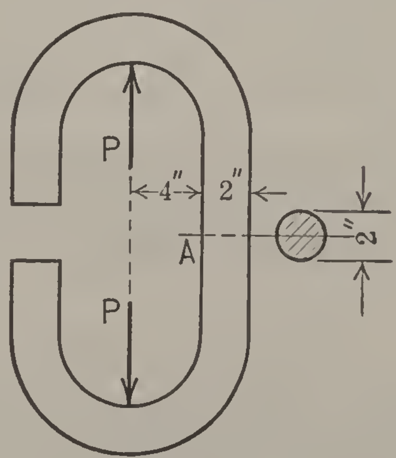


Fig. 170.

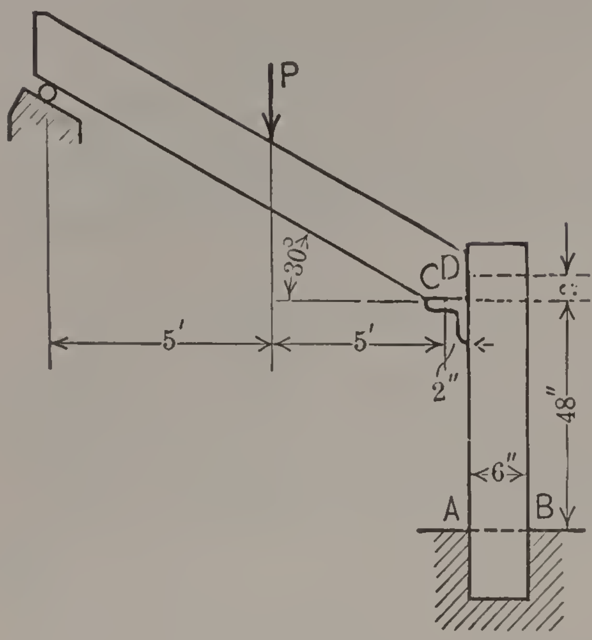


FIG. 171.

163. A timber post having a cross-section 6 in. by 6 in. supports one end of an inclined beam as shown in Fig. 171. The load P is 4000 lb. Find the unit-stress at points A and B of the section AB of the post.

88. Eccentric Load Not in Plane of Symmetry.—In the preceding article the action line of the resultant longitudinal load passed through a point on an axis of symmetry of the area on which the stress was desired. Now, as stated in Art. 43, in using the flexure formula $M=\frac{sI}{c}$ the neutral axis is assumed to be perpendicular to the plane of the loads, but this is true only when the loads lie in a plane of symmetry.[4]

If the load P (Fig. 172a) does not lie in a plane containing an axis of symmetry it may, as in the preceding article, be resolved into an axial load P_1 (equal to P) and a couple, P and P_2, having a moment Pe, and this couple may be further resolved into two component couples in planes containing the axes of symmetry or principal axes, OX and OY; the stress due to each component couple may then be found from the ordinary flexure formula. Thus, the moment of the component couple in the plane containing the axis OX is $Pe \cos \theta$ or Pe_x (Fig. 172a), the forces of this couple being represented by P_2 and P_3 in Fig. 172(b), and the moment of the component couple in the plane containing OY is Pe_y and the forces of this couple are P_4 and P_5. These five forces, each equal to P, will produce the same unit-stress at any point in a section, such as the section $ABCD$, as the original force

[4] The more general statement is that the loads must lie in a plane containing a principal axis of inertia of the cross-section; an axis of symmetry is always a principal axis; however, a section always has a principal axis even though it may not have an axis of symmetry.

P, and the normal unit-stress at any point in the section is the algebraic sum of the unit-stresses produced by the axial load P_1 and the two bending couples P_2, P_3, and P_4, P_5.

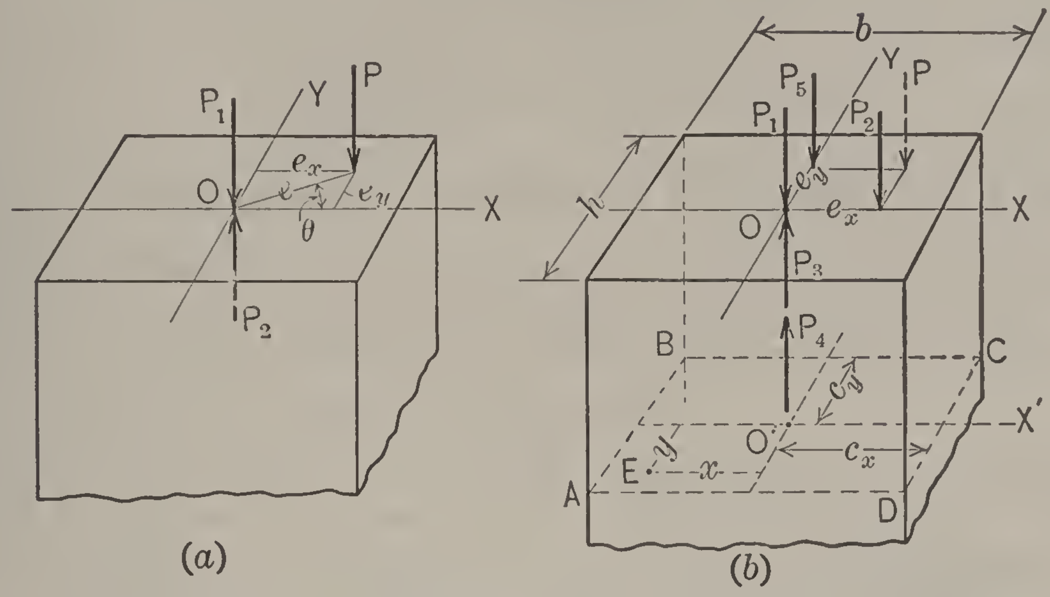


FIG. 172.—Eccentric load not in plane of symmetry.

The maximum unit-stress occurs at C and is

$$s=s_1+s_2+s_3$$

$$=\frac{P}{a}+\frac{Pe_xc_x}{I_y}+\frac{Pe_yc_y}{I_x}.$$

For other points in the area a either the second or the third term or both may be negative. Thus the unit-stress at a point E in the quadrant $O'A$ (Fig. 172b) is

$$s=\frac{P}{a}-\frac{Pe_xx}{I_y}-\frac{Pe_yy}{I_x}. \qquad (132)$$

Kern of a Section.—It is evident that the least value of s will occur at A and its value will be given by equation (132) when $x=\frac{b}{2}$ and $y=\frac{h}{2}$, and it will be a tensile stress if the sum of the last two terms is greater than $\frac{P}{a}$. If a tensile stress is to be avoided therefore, the values of e_x and e_y can not be greater than those found from the equation

$$\frac{Pe_x\frac{b}{2}}{I_y}+\frac{Pe_y\frac{h}{2}}{I_x}=\frac{P}{a},$$

or

$$\frac{e_x\frac{b}{2}}{\frac{1}{12}hb^3}+\frac{e_y\frac{h}{2}}{\frac{1}{12}bh^3}=\frac{1}{bh},$$

whence,

$$\frac{e_x}{b}+\frac{e_y}{h}=\frac{1}{6} \quad \text{or} \quad \frac{e_x}{\frac{1}{6}b}+\frac{e_y}{\frac{1}{6}h}=1,$$

which is the equation of a straight line that intersects the axis OX at a distance $\frac{b}{6}$ from O, and the axis OY at a distance $\frac{h}{6}$ from O (Fig. 173). Similar limits occur in the other quadrants and hence the resultant load on the member must pass within the shaded area shown in Fig. 173 if tensile stress in the member is to be avoided.

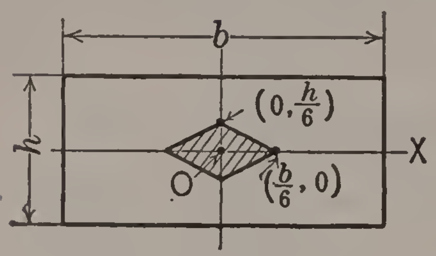


FIG. 173.—Kern of a section.

The area within which the resultant load must pass to avoid tensile stress on a section is frequently called the *core*, *kernel*, or *kern* of the section. Thus for a *rectangular section* the kern is a rhombus the diagonals of which are the middle-thirds of the principal axes of the section.

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164. A short rectangular timber having a section 6 in. by 8 in. is subjected to a longitudinal eccentric compressive load of 10,000 lb. The action line of the load passes through a point in each section 3 in. from the 6 in. side and $2\frac{1}{2}$ in. from the 8-in. side. Find the unit-stress at each corner.

Ans. $s_A=$ 56 lb. per sq. in. T., $s_B=156$ lb. per sq. in. C.
$s_C=468$ lb. per sq. in. C., $s_D=264$ lb. per sq. in. C.

165. The resultant normal pressure, P, on top of a concrete base having a square cross-section, a, acts at the center of one quadrant of the square. Will tensile stresses occur in the base? If so, find the maximum tensile and compressive unit-stress that will be developed, assuming the values of P and a to be 140,000 lb. and 4 sq. ft., respectively.

89. Eccentric Loads on Riveted Connections.—Examples of eccentric shearing loads are found frequently in riveted joints. Whenever feasible, however, eccentric loads should be avoided; that is, the action line of the resultant force to be transmitted through the rivets should pass through the centroid of the total rivet shearing area. Care in securing this condition, however, frequently is not given the attention it merits. The shearing stresses in riveted joints subjected to eccentric loads may be found as follows.

Let it be required to find the shearing unit-stress in each rivet in the riveted joint shown in Fig. 174(a). A force P having the eccentricity e is transmitted from the member A to the gusset plate B, and from the gusset plate through the four rivets to the member C. The shearing area of each rivet will be denoted by a, the diameter by d, and the distances of the centers of rivet areas from the centroid, G, of the total shearing area by l_1, l_2, etc.

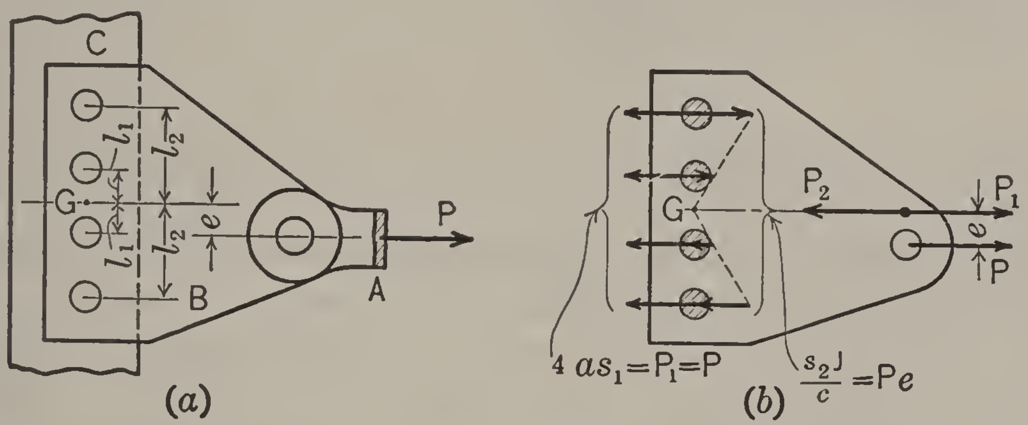

FIG. 174.—Shearing stress in rivets due to eccentric load.

The load P may be resolved into a load P_1 (Fig. 174b) acting through the centroid of the total shearing area and a twisting or torsional couple P_1, P_2 having the moment Pe. The central shearing load P_1 if acting alone would develop the same shearing unit-stress, s'_s, on each of the rivet areas (or rather this is the assumption usually made, see Art. 13).

The resisting stress that holds P_1 in equilibrium, then, is $4a\,s'_s$. Thus,

$$P_1=4as'_s \quad \text{or} \quad s'_s=\frac{P}{4a},$$

as indicated in Fig. 174(b).

And, if the couple Pe were acting alone it would develop a resisting moment $\left(\frac{s''_sJ}{c}, \text{ Art. } 26\right)$,[5] the shearing unit-stress in the

[5] In Art. 26 it was stated that $\frac{s_sJ}{c}$ is the expression for the resisting moment only for a solid or hollow cylindrical shaft. The rivet areas, however, may be transformed into annular areas having radii l_1 and l_2, similar to the cross-section of two concentric hollow cylindrical shafts.

rivets varying directly as the distances of the rivets from the centroid G, as shown in Fig. 174(*b*). This resisting moment holds the moment Pe in equilibrium. Hence,

$$Pe=\frac{s''_sJ}{c} \quad \text{or} \quad s''_s=\frac{Pec}{J}.$$

The shearing unit-stress on the bottom rivet, then, is

$$\begin{aligned} s_s &= s'_s+s''_s \\ &= \frac{P}{a}+\frac{Pec}{J}, \qquad \qquad (133) \end{aligned}$$

in which c is the distance from G to the rivet on which the stress is desired (equal to l_2 for the bottom rivet), and J is the polar moment of inertia of the total shearing area of the rivets with respect to an axis passing through G.

Since the diameters of the rivet areas are small in comparison with the distances of the rivets from G, the value of J is, with slight error, the sum of the products obtained by multiplying each rivet area by the square of the distance of its center from G. Thus,

$$J=2a(l_1{}^2+l_2{}^2)$$

and hence

$$Pe=\frac{2as''_s}{l_2}(l_1{}^2+l_2{}^2). \qquad \qquad (134)$$

Instead of obtaining the resisting moment from the expression $\frac{s''_sJ}{c}$, it may also be found as the sum of the moments, with respect to G, of the total shearing stresses on the rivet areas. Thus if s''_s denotes the shearing unit-stress on the rivets farthest from G, and s''_{s1} on the rivets nearest to G, we have

$$\begin{aligned} Pe &= 2[as''_sl_2+as'_{s1}l_1] \\ &= 2\left[as''_sl_2+a\left(\frac{l_1}{l_2}s''_s\right)l_1\right], \text{ since } s'_{s1}=\frac{l_1}{l_2}s''_s \\ &= 2\frac{as''_s}{l_2}(l_1{}^2+l_2{}^2), \end{aligned}$$

which is the same as equation (134).

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166. The diameter of the rivets in the joint shown in Fig. 175 is $\frac{3}{4}$ in., and the load P is 4000 lb. Find the shearing unit-stress in the top and bottom rivets.
Ans. $s = 6750$ lb. per sq. in.

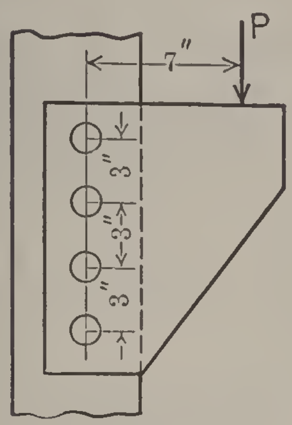

Fig. 175.

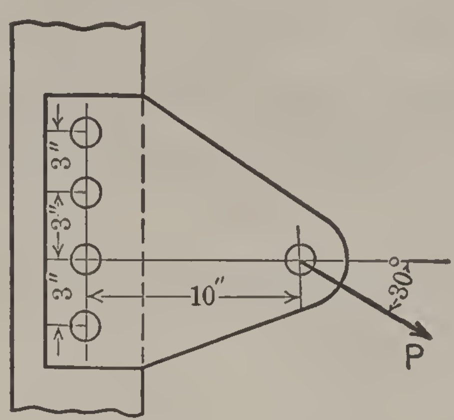

Fig. 176.

167. The load P transmitted through the joint shown in Fig. 176 is 5000 lb. and the diameter of the rivets is $\frac{3}{4}$ in. Find the shearing unit-stress in the top and in the bottom rivet.

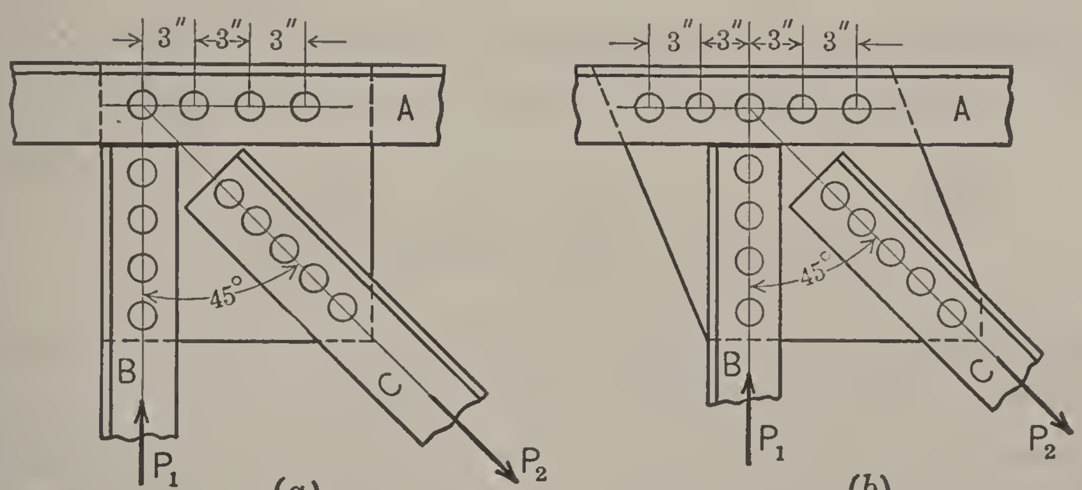

Fig. 177.

168. Fig. 177(a) shows a commonly used joint in a riveted Pratt truss. If the joint is arranged as shown in Fig. 177(b) how much will the maximum shearing unit-stress in the rivets in the top chord A be decreased? All the rivets have a diameter of $\frac{3}{4}$ in. The values of P_1 and P_2 are such as to produce a shearing unit-stress of 10,000 lb. per sq. in. in the rivets connecting the two members, B and C, to the gusset plate. (See Prob. 38.)

90. Helical Spring. *Stress Developed.*—A helical or coil spring (Fig. 178a) is frequently used to resist an axial load P that lengthens or shortens the spring. The stress developed in the rod of which the spring is made is mainly torsional shearing stress.

The value of the shearing unit-stress in a spring made of rod having a *circular cross-section* may be found as follows:

Let a section be passed through the rod at A (Fig. 178a) and the lower part of the spring be removed. The part above the section A is shown in Fig. 178(b). Further, let the load P be resolved into a force P_1 (whose action line passes through the center of the section at A) and a twisting moment Pe, by introducing the two forces P_1 and P_2 (Fig. 178b) each equal to P. Now these forces

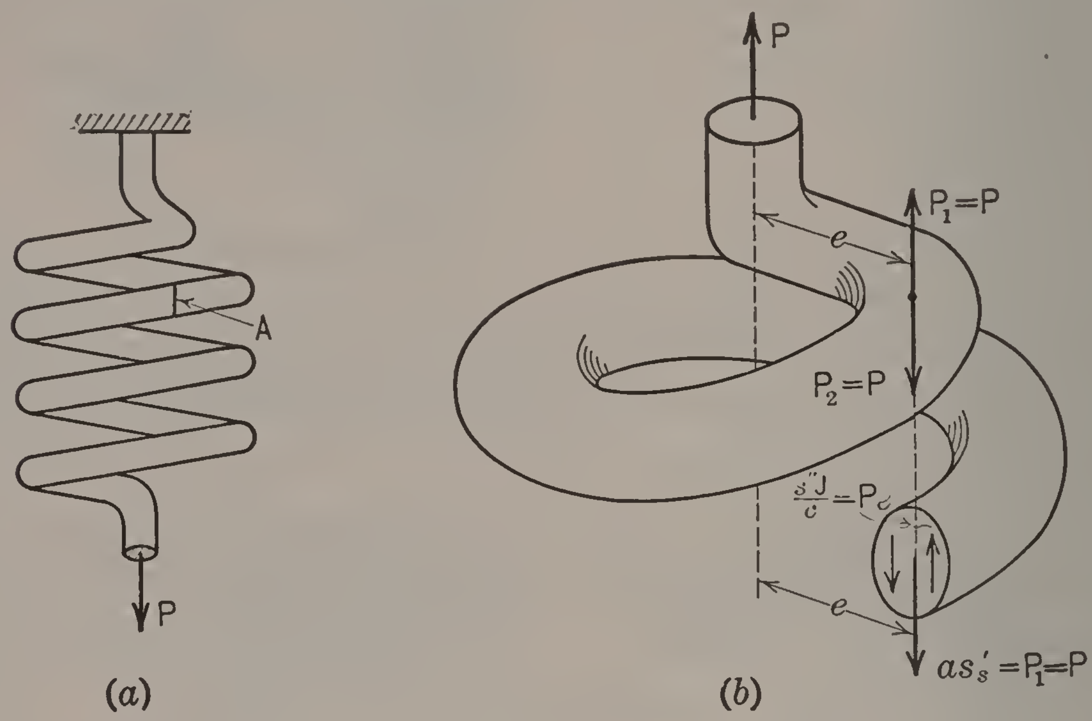

FIG. 178.—Stress in helical spring.

(which produce the same stress on the section at A as does the original one force P) are held in equilibrium by the stresses on the section at A. If the force P_1 were acting alone it would produce a resisting shearing stress that would hold P_1 in equilibrium, and this direct shearing stress would be equal to as'_s, as indicated in Fig. 178(b), a being the area of cross-section of the rod, and s'_s the shearing unit-stress. Hence,

$$P_1=P=as'_s \quad \text{or} \quad s'_s=\frac{P}{a}.$$

Further, if the twisting moment Pe due to the couple P, P_2 were acting alone it would produce a resisting moment $\frac{s''_s J}{c}$ on the area a which would hold the external or twisting moment in equilibrium as indicated in Fig. 178(b). Hence

$$Pe=\frac{s''_s J}{c}.$$

Now the direct shearing unit-stress s'_s in most springs is negligible in comparison with the torsional unit-stress s''_s, and hence the chief unit-stress in the rod is due to torsion and its maximum value is

$$s''_s = \frac{Pec}{J},$$

in which e is the mean radius of the coil, c the radius of the rod $\left(\text{equal to } \frac{d}{2} \text{ where } d \text{ is the diameter of the rod}\right)$ and J is the polar moment of inertia of the cross-sectional area of the rod $\left(J = \frac{\pi d^4}{32},\right.$ Art. 162).

The above expression then may be written

$$s''_s = \frac{Pe\frac{d}{2}}{\frac{\pi d^4}{32}} = \frac{16Pe}{\pi d^3}.$$

If, however, the direct unit-stress s'_s is not negligible, the maximum unit-stress on the area a will be the sum of s'_s and s''_s. Thus

$$s_s = \frac{P}{a} + \frac{16Pe}{\pi d^3}$$

$$= \frac{P}{a}\left(1 + \frac{4e}{d}\right). \qquad (135)$$

Deflection of Helical Spring.—The increase or decrease in length (called deflection) of the spring due to a given load P may be found as follows:

If the shearing proportional limit of the material is not exceeded, the work done by the load, as it increases uniformly from zero to the value P, is stored in the material as stress (potential) energy, and further, if the stress developed is assumed to be due only to torsion, it follows that

Work done in causing deflection = Work done in twisting the rod.

Now the work done by the load in deflecting the spring is the product of the average value of the load $\left(\frac{P}{2}\right)$ and the deflection Δ. Similarly, the work done in twisting the rod through an angle θ is the product of the average value of the twisting moment $\left(\frac{T}{2} \text{ or } \frac{Pe}{2}\right)$ and the angle of twist. Hence,

$$\tfrac{1}{2}P\Delta = \tfrac{1}{2}T\theta = \tfrac{1}{2}Pe\theta.$$

Therefore,

$$\Delta = e\theta.$$

But from Art. 28, the value of θ, is given by the equation,

$$\theta = \frac{Pel}{E_sJ}.$$

Further, for a closely coiled spring, $l = 2\pi en$, approximately, where n is the number of turns. Hence,

$$\Delta = e\frac{Pe2\pi en}{E_sJ}, \qquad (136)$$

but, since $J = \frac{\pi d^4}{32}$, the above expression becomes,

$$\Delta = \frac{64Pe^3n}{E_sd^4}. \qquad (137)$$

If Δ is expressed in inches and P in pounds, then e and d are expressed in inches, and E_s in pounds per square inch. If the shearing unit-stress developed in the bar is greater than the shearing proportional limit the above equation is not valid.

The deflection, Δ_s, due to the direct shear is $\Delta_s = \frac{Pl}{aE_s}$ and hence the total deflection is

$$\text{total deflection} = \frac{Pl}{aE_s} + \frac{Pe^2l}{E_sJ} = \frac{Pl}{aE_s}\left(1 + \frac{e^2}{k^2}\right), \qquad (138)$$

in which k is the polar radius of gyration of the section.

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169. A helical spring when compressed 4 in. must support an axial load of 1600 lb. The diameter of the rod is $\frac{5}{8}$ in., the mean diameter of the coil is 3 in. and the working shearing unit-stress of the material (spring steel), is 60,000 lb. per sq. in. Find the number of coils required. Is the working stress exceeded? *Ans.* $n = 21.2$ coils. $s_s = 50{,}000$ lb. per sq. in.

170. A helical spring made of steel rod $\frac{1}{4}$ in. in diameter with 5 turns is 2.65 in. high and has an outside diameter of 1.35 in. Find the modulus of the spring, that is, the force required to stretch it an inch, assuming that the proportional limit of the material is not exceeded.

Ans. Mod. = 965 lb., if deflection due to direct shear is neglected.

171. A bronze helical spring is required to support 980 lb. when deflected 3 in. Find a suitable diameter of the coil and number of turns if the diameter of the rod is $\frac{1}{2}$ in. and the working unit-stress not to exceed 50,000 lb. per sq. in. $E_s = 5{,}000{,}000$ lb. per sq. in.

CHAPTER XI

COMPRESSION MEMBERS. COLUMNS

91. Introduction.—A compression member is a bar (either solid or built-up, and, as a rule, approximately straight) subjected to an end-load (or loads) acting parallel to the axis of the bar. If the load acts through the centroid of the end section it is called an axial load; if not, it is called an eccentric load. But, an eccentric load is equivalent to an axial load and an end-moment or couple. The character of end-loads applied to compression members as used in engineering structures and machines depends largely on the type of end-connection used in the structure, and on the relative stiffness of the members connected. For example, a locomotive connecting rod, many members in certain types of bridges, etc., are connected by pins, and the end-load exerted on a compression member by the pin is mainly a single force, since the end-moment due to the friction of the pin is relatively small; whereas, most members in buildings, etc., are connected by rivets either directly or by means of an auxiliary piece such as a gusset plate, and a compression member may be subjected to an end-moment, in addition to a compressive load, due to the bending of a member (a beam, say) to which the compression member is attached.

If a compression member is slender (relatively long), it is usually called a column; and if relatively short it is called a compression block or a strut. This chapter deals mainly with the strength of columns; there is, however, no definite slenderness that differentiates a column from a compression block.

In the investigation of the stresses and deformations in a column there must be considered, as affecting the action of the column, certain conditions that may be neglected in the investigation of the stresses and deformations in most other structural members. For example, in determining the stresses and deformations in beams, tension members, torsional shafts, etc., when resisting loads, the member in question is considered to satisfy ideal conditions; namely, the body is considered to be straight, to be made of homogeneous material free from initial stresses, to be so constructed that it acts as a whole or as a unit, and to be subjected to loads having definitely known action lines. And, although it is known that such ideal conditions are never fully realized in these structural members, experiments and experience have shown that, in general, the results found from an analysis of the actions in the ideal beam, tension member, etc., are reliable; that is, the primary or significant action in the actual member is the same as that in the ideal body. But, the action in an actual column, as shown by experiments and experience, may deviate much from that in an ideal column. This fact may be stated in a somewhat different way as follows: In beams, tension members, etc., conditions that differ from those in the ideal members do not have more than a secondary effect on the stresses and deformations in the beams, tension members, etc., whereas the similar conditions in a column may be of prime importance in determining the resistance of the column.

Further, a quantitative measure of the effects of these conditions cannot, in general, be determined, and hence they render the analysis of column action less definite and satisfactory than that of beams, tension members, etc., and at the same time, cause a column to be a more critical member in a structure than most other structural members.

A brief discussion of the actions and conditions that must be especially considered in a column, are given in the next article and also in Art. 97.

92. Distinguishing Features of Column Action.—The fundamental idea of column action is that of combined compression and bending, but the bending action in a column is accompanied by certain conditions that renders a quantitative measure of the bending action less certain than that of the bending in a beam or in a member subjected to combined tension and bending, etc. The distinguishing features of column action may be stated briefly as follows:

1. A straight column may bend or deflect as a whole when subjected to a supposedly axial end-load (Fig. 179*a*), the initial bending being due to a slight initial or accidental eccentricity[1] of load-

[1] If the column bends there must be a bending moment to start the bend-

ing, and this deflection causes an increase in the moment-arm of the load, and hence increases the bending moment; the increased bending moment in turn increases the deflection, etc., tending to cause failure of the column by flexure or what is frequently called "buckling." This action is of special importance in the more slender columns and in the component parts of built-up columns.

In a column, then, the bending moment may increase more rapidly than the load increases, whereas this undesirable condition does not occur in other structural members. For example, in a straight compression block the deflection may be assumed to be negligible and hence the initial bending moment, if any, due to initial eccentricity, etc., increases only as the load increases since the moment-arm of the load remains constant. Likewise, in the case of a beam, although the deflection of the beam may be greater than that of a column, the deflection has a negligible effect on the moment-arm of the load. Again, in the case of a tension member subjected to an eccentric load, the bending of the member causes the moment-arm of the load to decrease, and hence the bending moment increases even less rapidly than the load increases.

2. By virtue of the action discussed under (1) above, an initial crookedness in a column as a whole (and all columns are more or less crooked) reduces the strength of a column more than a like crookedness reduces the strength of a beam, tension member, etc.

3. Local kinks or bends in the component parts of a built-up column, lack of homogeneity of material, initial stresses [2] in the material, and poorly designed end-connections, may cause stresses and deformations in parts of a column greatly in excess of those assumed to exist in the corresponding ideal column, and these localized stresses and deformations, as a rule, reduce the resistance of the column more than similar localized stresses reduce the resistance of beams, tension member, etc.

4. A variation in the position of the action line of a load on a column from that assumed in the ideal column, that is, the presence of accidental or fortuitous eccentricities, cause, in general, a much greater increase in the stresses in a column than similar variations cause in the stresses in beams, in tension members, etc.

ing; various conditions contributing to this bending are discussed later but an initial or accidental eccentricity may here be considered to be the chief cause of the initial bending.

[2] For causes of initial stresses see Art. 106 and 137.

Further, such fortuitous eccentricities are very likely to occur due to the actions of the end-connections.

93. Slenderness Ratio.—The slenderness ratio for a column is a term that enters in all column formulas; it is the ratio of the length, l, of the column to the radius of gyration, r, of the cross-section with respect to the neutral axis, the neutral axis being the centroidal axis that is perpendicular to the plane in which the column bends or tends to bend. If the column is pivot-ended, that is, free to turn at its ends, it will bend in a plane perpendicular to the axis of the cross-section about which the moment of inertia is least, and r will be the *least* radius of gyration. Column sections are frequently designed so that the moments of inertia of the section about the principal axes are approximately equal; this condition, however, is not always desirable.

94. Two Limiting Cases of Compression Members.—If a short solid compression block is subjected to an end load having a very slight eccentricity, the block may be assumed to be in pure compression since the bending action may be neglected. But if this block be assumed gradually to become longer (more slender) the tendency to bend becomes greater, and the bending cannot be considered to be negligible if the length of the member is greater than about ten times the least lateral dimension; that is, if the slenderness ratio is about 40. And, if the column is very slender, the resistance of the column is mainly its resistance to bending, the effect of the direct compression then being negligible. The slenderness ratios of compression members used in most engineering structures and machines, however, are such that both direct compression and bending are of importance, and such members will be discussed after considering the two limiting cases of compression members.

One limiting case of a compression member, then, is a compression block, and for such a member, when made of homogeneous material and subjected to an axial load P, the formula is

$$P = as,$$

in which a is the cross-sectional area of the column and s is the unit-stress at all[3] points in the area.

[3] The variation of stresses over the cross-section in short *built-up* columns is discussed in Art. 106 and 137.

The other limiting case is that of a member so slender that bending or flexure is the only action that need be considered, and hence the flexure formula, $M=\frac{sI}{c}$, for such a column applies, the bending moment M at any section being Py. (Fig. 179a). Thus,

$$Py=\frac{sI}{c},$$

which states that the bending moment (Py) of the external forces about any section is held in equilibrium by the resisting moment $\left(\frac{sI}{c}\right)$ of the stresses developed on that section. But the value of the deflection y in the equation corresponding to a given load P is unknown and hence the value of P that produces a given stress s cannot be found, since there are two unknown quantities in the equation; in other words, the problem is statically indeterminate; but, as discussed in Chapter VII, the elastic curve equation offers another method for finding external forces. The expression for the load that causes a slender column to bend as found from the elastic curve of the column was first obtained by Euler in 1757 and is known as Euler's column formula; it is derived in the following article.

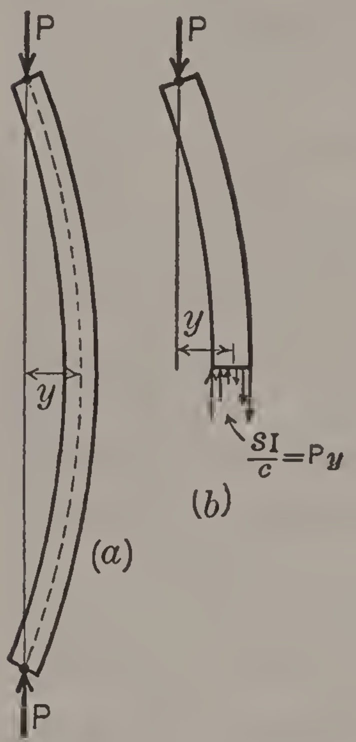

Fig. 179.—Bending of slender column.

95. Euler's Column Formula.—The problem is to find the *least value of an axial load P that will cause a column to bend.* The importance of this value of the load lies in the fact that, due to the conditions stated in Art. 92, a column fails by bending if the load is increased only a small amount above this value, and hence the least value of the load that causes a column to bend *is also the maximum load the column can resist,* unless the column fails by crushing of the material by direct compression before this value is reached.

Now the method of obtaining this critical bending load is to assume that an ideal column is deflected by a lateral force [4] and that an axial load P is applied that will maintain this deflection

[4] An axial load could not cause an *ideal* column to bend.

after the lateral force is removed; a relation between the load and the deflection is then found from the elastic curve equation of the deflected column, and the limiting value of P found from the equation, by expressing the fact that the deflection is indefinitely small, is the least value of P that will cause an ideal column to bend (or the maximum value that will not cause bending).

In Fig. 180(a) is represented a slender solid column of constant cross-section; it is assumed to be straight, to be made of homogeneous material free from initial stresses, and to be subjected to an axial end-load only, that is, no restraints occur at the ends and the weight of the column is neglected. Let the load P be such as to hold the column in the deflected position as shown in Fig. 180(b). Now since bending is the only action here considered the equation of the elastic curve is the same as that of a beam; namely (see Art. 45),

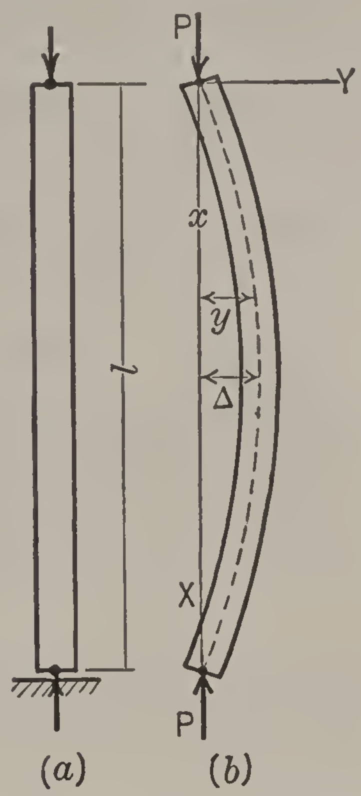

FIG. 180. — Elastic curve of slender column.

$$\pm EI\frac{d^2y}{dx^2}=M,$$

in which M is positive and equal to Py, and, if the origin of axes is chosen at the upper end of the column and the positive direction of the x and y axes are chosen downward and to the right, respectively, as shown in Fig. 180(b), then $\frac{d^2y}{dx^2}$ is negative (see Art. 45), and the above equation becomes

$$EI\frac{d^2y}{dx^2}=-Py. \qquad (139)$$

This equation may be written

$$EI\frac{d\left(\frac{dy}{dx}\right)}{dx}=-Py, \qquad (139a)$$

and if each side of the equation is multiplied by $\frac{dy}{dx}$, each side becomes an exact derivative of the form udu; namely,

$$EI\left(\frac{dy}{dx}\right)d\left(\frac{dy}{dx}\right)=-Pydy.$$

Integrating, we have,

$$EI\left(\frac{dy}{dx}\right)^2=-Py^2+C_1. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (140)$$

Now simultaneous values of $\frac{dy}{dx}$ and y are not known, and hence C_1 cannot be expressed in terms of the known quantities E, I and l, but it may be expressed in terms of the maximum deflection, Δ. Thus, since $\frac{dy}{dx}=0$ when $y=\Delta$, we have

$$0=-P\Delta^2+C_1; \quad \text{whence} \quad C_1=P\Delta^2,$$

and eq. (140) becomes

$$EI\left(\frac{dy}{dx}\right)^2=P(\Delta^2-y^2). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (141)$$

This equation may be integrated after the variables are separated; the equation may be written as follows:

$$\frac{dy}{\sqrt{\Delta^2-y^2}}=\sqrt{\frac{P}{EI}}dx.$$

Integrating, we have

$$\sin^{-1}\frac{y}{\Delta}=\sqrt{\frac{P}{EI}}x+C_2, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (142)$$

or

$$\frac{y}{\Delta}=\sin\left(\sqrt{\frac{P}{EI}}x+C_2\right).$$

But $y=0$ when $x=0$ and hence $C_2=0$. Therefore,[5]

$$\frac{y}{\Delta}=\sin\sqrt{\frac{P}{EI}}x \quad \text{or} \quad y=\Delta\sin\sqrt{\frac{P}{EI}}x. \quad . \quad . \quad . \quad . \qquad (143)$$

[5] If the origin of axes is chosen at the center of the column as in Fig. 181, eq. (143) becomes

$$y=\Delta\cos\sqrt{\frac{P}{EI}}x.$$

This may be shown as follows: The above derivation as far as eq. (142) is

Now if the deflection of a column is considered to decrease, the value of P required to maintain the deflection also decreases and attains its limiting (minimum) value when the deflection becomes indefinitely small, and since the column is then virtually straight, the vertical distance, x, between the ends of the column is equal to l. Therefore, the limiting value of P may be found by substituting the values $y=0$ when $x=l$ in eq. 143. Whence,

$$\sin\left(\sqrt{\frac{P}{EI}}\cdot l\right)=0,$$

and hence the angle $\left(\sqrt{\frac{P}{EI}}\cdot l\right)$ must be equal to π or some integral multiple of π; as discussed in Art. 100, the value is π for a column with pivoted ends and 4π for one with fixed ends, etc. Therefore, the minimum value of P for a column having pivoted ends is

$$P=\frac{\pi^2 EI}{l^2}. \qquad (144)$$

An axial load P, then, less than that given by eq. (144) will not cause a column to bend, whereas a load greater than this will cause an *actual* column to bend, or will maintain an *ideal* column in a deflected position. And, although an ideal slender column may resist a somewhat larger load than this, the accompanying deflec-

independent of the origin of axes. Now referring to Fig. 181 and eq. (142), we have, $y=\Delta$ when $x=0$, and hence

$$\sin^{-1}1=0+C_2, \quad \text{whence} \quad C_2=\frac{\pi}{2}.$$

Therefore eq. (142) becomes

$$\sin^{-1}\frac{y}{\Delta}=\sqrt{\frac{P}{EI}}x+\frac{\pi}{2},$$

or

$$\frac{y}{\Delta}=\sin\left(\frac{\pi}{2}+\sqrt{\frac{P}{EI}}x\right),$$

whence,

$$y=\Delta\cos\sqrt{\frac{P}{EI}}x.$$

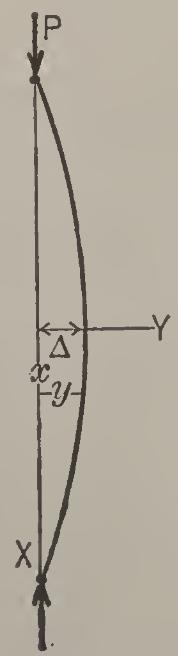

FIG. 181.

tion becomes excessive [6] when P has a value only slightly greater than that given by eq. (144). Therefore, the value of P in eq. (144) is considered to be the maximum load an actual (structural) column can resist without *failing by bending*. It may fail, however, by crushing before this value is reached, as will be discussed later.

The fact that excessive deflections occur when P has a value close to the Euler load is clearly shown by testing a slender column, such as a wooden airplane strut, and measuring the deflection. The load-deflection curve obtained is shown in Fig. 182; the deflection starts somewhat before the Euler load is reached due to slight eccentricities, etc., but at the Euler load the deflection increases rapidly to a value equal to AB with very slight increase of load, the stress in the column being less than the proportional limit until the deflection AB is exceeded.

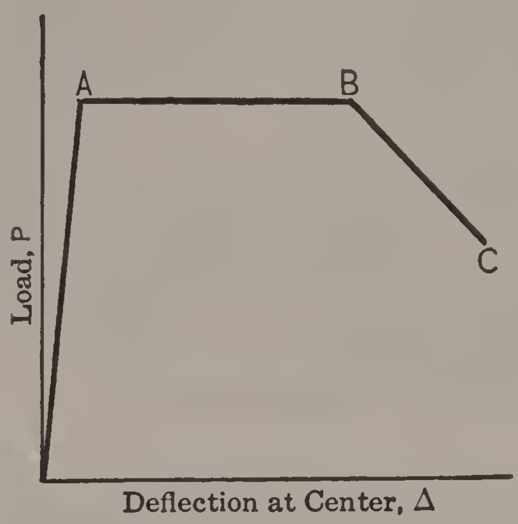

FIG. 182.—Relation between load and deflection for slender column.

Since I in eq. (144) is equal to ar^2 (Art. 163) where a is the area of cross-section and r is the radius of gyration of the cross-section about the neutral axis, Euler's equation may be written

$$\frac{P}{a}=\frac{\pi^2 E}{\left(\frac{l}{r}\right)^2}, \qquad (145)$$

in which $\frac{l}{r}$ is the slenderness ratio of the column.

Eq. (144) shows that the only property of the material on which P depends is the stiffness of the material, E (see Art. 144 and 145); therefore, a column made of high-carbon or special alloy steel would begin to bend, and hence fail, at the same load as would

[6] This fact is not evident from the analysis of the column action here presented; it is shown by the true equation of the elastic curve, that is, the equation obtained when the assumption that the length along the axis is the same as the length along the curve ($dx=dl$) is not made. The truth of the statement, however, is clearly shown by experiments as indicated by the results represented in Fig. 182.

one made of low-carbon steel, provided that the $\frac{l}{r}$ were relatively large so that the columns would not fail by crushing before the Euler load is reached.

ILLUSTRATIVE PROBLEM

Problem 172. A column made of a 10-in., 35-lb. I-beam and two plates $\frac{5}{8}$ in. by 11 in., as shown in Fig. 183, is 32 ft. long and is subjected, in a testing machine, to an axial load applied through spherical-seated bearings. Calculate the least radius of gyration, r, and the corresponding slenderness ratio, $\frac{l}{r}$. Find the maximum load the column can resist, and also a working load using a reduction factor (so-called factor of safety) of 3.

Solution.—The moment of inertia of the cross-sectional area about the x-axis may be found as follows: Area of each plate, $a_1 = 6.87$ sq. in. A steel maker's handbook gives:

Area of the I-section $= 10.29$ sq. in.
I_x of the I-section $= 146.4$ in.^{4}
I_y of the I-section $= 8.5$ in.^{4}

Hence,

Total area of cross-section, $a = 24.03$ sq. in.

I_x of the two plates $= 2(\bar{I} + a_1 d^2)$, (see Art. 164)
$= 2\, a_1 d^2$ approximately
$= 2 \cdot 6.87 \cdot (5.3)^2$
$= 386$ in.^{4}.

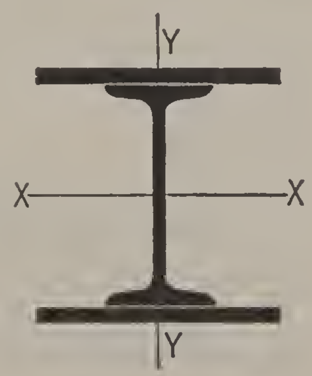

FIG. 183.

$$I_x \text{ for whole area} = 146.4 + 386 = 532.4 \text{ in.}^4$$
$$I_y \text{ for whole area} = 8.5 + 2(\tfrac{1}{12}bh^3) = 8.5 + 2(\tfrac{1}{12}\,\tfrac{5}{8})(11)^3 = 8.5 + 138.6 = 147.1 \text{ in.}^4$$

Since I_y is smaller than I_x the column will bend so that the y-axis is the neutral axis, that is, in a plane perpendicular to the y-axis.

The least radius of gyration, $r = \sqrt{\frac{I_y}{a}} = \sqrt{\frac{147.1}{24.03}} = 2.47$ in. Therefore,

$$\frac{l}{r} = \frac{32 \times 12}{2.47} = 155.5,$$

and this value is sufficiently large, as discussed in the next article, to allow the column to bend before failing by direct compression.

Hence,

$$\text{Max. unit-load} = \frac{P}{a} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2} = \frac{9.87 \times 30{,}000{,}000}{(155.5)^2} = 12{,}250 \text{ lb. per sq. in.}$$

$$P = 12{,}250 \times 24.03 = 294{,}200 \text{ lb.}$$

$$\text{Working load} = \frac{294{,}200}{3} = 98{,}100 \text{ lb.}$$

96. Graphical Representation of Formulas for Ideal Columns.—As already noted, the maximum axial load P that an ideal column having a very small slenderness ratio (a compression block) can resist depends on the compressive strength, s, of the material and the cross-sectional area a the equation being $\frac{P}{a} = s$, whereas the maximum load that a slender column can resist depends on the stiffness, E (not the strength), of the material and on the slenderness ratio $\frac{l}{r}$, in addition to the area a; the equation being $\frac{P}{a} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2}$.

Now the results of tests of axially loaded columns agree approximately with the calculated values from the above equations when the columns have small and large values, respectively, of $\frac{l}{r}$. Thus, if a series of ideal columns were made of one kind of material but with different slenderness ratios, and were tested to failure by applying axial loads, the columns having small values of $\frac{l}{r}$ (less than about 40) would fail by direct compression when the compressive unit-stress in the material reached the ultimate compressive strength of the material; the maximum useable compressive strength of ductile material in columns, however, is the yield point of the material since at this stress plastic flow of the material occurs and the column fails to perform its structural function even if total collapse does not result. But the columns having large values of $\frac{l}{r}$ would fail by buckling when the unit-load

$\frac{P}{a}$ is slightly greater than that given by Euler's formula; the unit-stress in the column when the buckling starts being less than the proportional limit of the material.

If, then, values of $\frac{P}{a}$ that cause failure are plotted (Fig. 184) as ordinates and the corresponding values of $\frac{l}{r}$ as abscissae the column formula $\frac{P}{a}=s$ is represented by the horizontal line AB, and Euler's equation is represented by the curve CBD.

An ideal column having an $\frac{l}{r}$ greater than that represented by the abscissa OB' (Fig. 184) fails by bending, and an ideal column having a slenderness ratio less than this critical value fails by the yielding of the material in direct compression if the material is ductile, or by crushing or diagonal shearing of the material if the material is brittle.

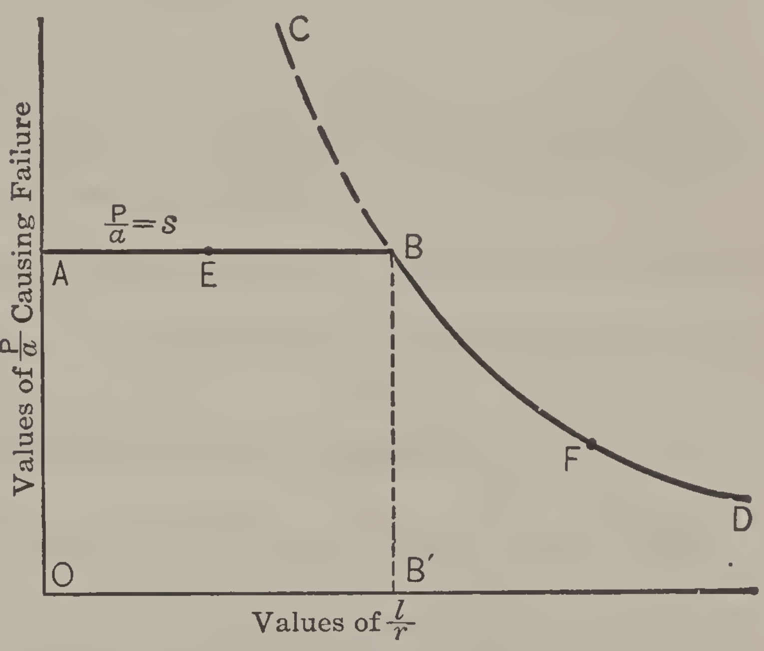

FIG. 184.—Graphical representation of ideal-column formulas.

The graph representing the column formulas for ideal columns is, therefore, the curve ABD of Fig. 184; the part BC of the Euler curve has no physical meaning, since the column would fail by crushing before a load represented by an ordinate to BC could be applied.

ILLUSTRATIVE PROBLEM

Problem 173. Find the critical value of $\frac{l}{r}$, corresponding to OB' in Fig. 184, for an axially-loaded pivot-ended ideal structural-steel column.

Solution.—The yield point of structural steel is about 40,000 lb. per sq. in.

and $E=30,000,000$ lb. per sq. in. Now $\frac{P}{a}$ as found from the two column formulas are equal when $\frac{l}{r}$ has the value OB' (Fig. 184), and hence,

$$\frac{P}{a}=s=\frac{\pi^2 E}{\left(\frac{l}{r}\right)^2}.$$

That is,

$$40,000=\frac{\pi^2\times 30,000,000}{\left(\frac{l}{r}\right)^2},$$

whence,

$$\frac{l}{r}=86.$$

Therefore, an ideal axially-loaded pivot-ended structural-steel column would fail by buckling if its slenderness ratio were greater than about 86. Tests of actual pin-ended steel columns, however, show that Euler's equation is applicable only when $\frac{l}{r}$ is greater than about 120, for reasons discussed in Art. 92 and in the following articles.

PROBLEMS FOR ARTICLES 95 AND 96

174. Find the least value of $\frac{l}{r}$ for an ideal pivot-ended oak column that will fail by bending when subjected to an axial load. Assume the following values for the ultimate compressive strength s_u, and modulus of elasticity: $s_u=8000$ lb. per sq. in., $E=1,500,000$ lb per sq. in.

175. A 3-in. by 3-in. by $\frac{1}{2}$ in. wrought iron angle 15 ft. long was tested by the Pencoyd Steel Co., spherical seated bearings at the ends being used. It failed when subjected to an axial unit-load of 2650 lb. per sq. in. What is the $\frac{l}{r}$ for the column? Calculate the maximum unit-load the column would be expected to resist. ($E=25,000,000$ lb. per sq. in.)

Ans. $\frac{l}{r}=310$; $\frac{P}{a}=2560$ lb. per sq. in.

176. Design by Euler's formula, a square pivot-ended timber column 20 ft. long to resist an axial load of 60,000 lb. Use a working load equal to $\frac{1}{8}$ of the given load; assume $E=1,500,000$ lb. per sq. in.

177. What is the ratio of the strength of an ideal slender solid cast-iron column 6 in. in diameter to the strength of a slender hollow cast-iron column having a wall thickness of 1 in., the areas of cross-section and the lengths of the two columns being equal? *Ans.* 9 to 41.

97. Methods of Obtaining Formulas for Columns Having Intermediate Slenderness Ratios.—The results of many tests of columns show that the formulas represented in Fig. 184 are substantially correct for small and large values, respectively, of $\frac{l}{r}$ but for intermediate values of $\frac{l}{r}$ (corresponding to the part of the curve between E and F say) the test results for $\frac{P}{a}$ fall below the curve, the cause of the lower values of $\frac{P}{a}$ being due to the conditions stated in Art. 92. Now the columns used in engineering structures and machines have these intermediate values of $\frac{l}{r}$, and the conditions stated in Art. 92 have, in general, a relatively greater influence on the strength of columns with intermediate slenderness ratios than on columns with either small or large values of $\frac{l}{r}$. Further, as noted in Art. 91, it is practically impossible to obtain a rational column formula that gives quantitative expression to the influences of these conditions, and hence formulas for columns of intermediate slenderness ratios are always either partly or entirely empirical. The main methods that have been used for developing such formulas are as follows:

1. To assume that the conditions stated under (2), (3) and (4) of Art. 92 will cause the column to bend as a whole when subjected to an axial load P, the maximum bending moment, at the center of the column, being $P\Delta$, where Δ is the deflection of the column. The maximum stress in the column is considered to be the sum of the direct compressive stress and the bending stress due to the moment $P\Delta$. The value of the deflection, being unknown, is eliminated from the equation by assuming that the relation between the stress and deflection in the column follows the same law as does stress and deflection in a beam. The widely used Gordon-Rankine formula, discussed in Art. 98, is obtained by this method; it must be regarded as essentially an empirical formula, for, although the statement that the stress in the column, is due to combined direct compression and bending is essentially correct, the assumption made as to the relation between stress and deflection is untenable; and further, the quantitative measure of the

influence of the conditions stated in Art. 92 on the stress in the column is determined from experimental results.

2. To determine from test data an equation which respresents the average of test results without any attempt at a rational analysis of column action. This method gives purely an empirical formula; the widely used straight-line formula discussed in Art. 99 is obtained in this way.

3. To assume that the combined effects of the conditions stated in Art. 92 are equivalent, in producing stress, to a eccentricity, e, of loading, called an *equivalent* eccentricity. The bending moment, then, after the column bends is $P(e+\Delta)$ instead of $P\Delta$ as used in Euler's equation. An expression for P is then found, from the elastic curve equation, which contains the maximum unit-stress in the column and the equivalent eccentricity, e; a value for e being chosen so that the formula is made to agree with experimental results. This method leads to a modified Euler formula called the secant formula which is discussed in Art. 105.

98. Gordon-Rankine Formula.—As already noted, an ideal column subjected to an axial load would not bend until the Euler load is reached, but the deviations from these ideal conditions may be assumed to be equivalent, in producing stress, to an initial crookedness or bend in the column as a whole, the total deflection at the center after an axial load P is applied being Δ (Fig. 185a). The unit-stress, then, in the column is due to direct compression and bending.

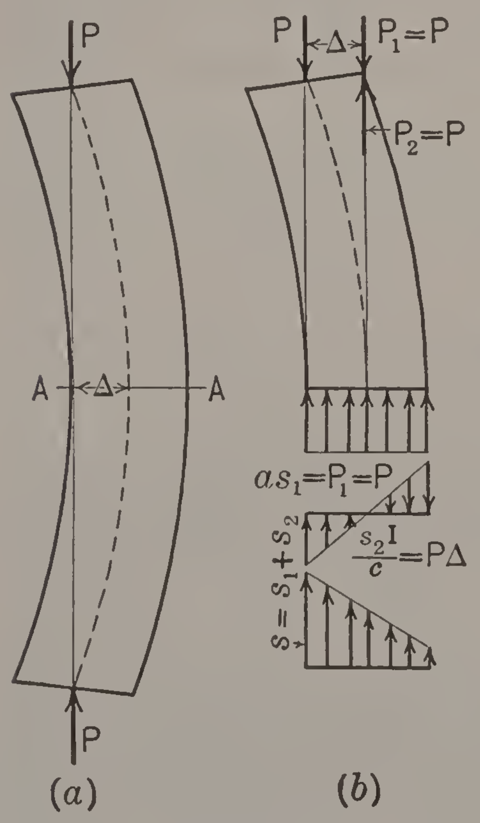

FIG. 185.—Stress in Column.

The maximum value of the unit-stress is found as follows: The maximum stress occurs on section AA (Fig. 185a) and the stresses on this section must hold the external force P in equilibrium. Now, by introducing two equal and opposite forces, P_1 and P_2, each equal to P, the force P may be resolved into a force P_1 (Fig. 185b) acting through the centroid of the central section AA, and a bending couple $P\Delta$. If the force P_1 were acting alone the unit-stress s_1 developed on the area

a of the section AA would be uniformly distributed over the area, and hence

$$as_1 = P, \quad \text{or} \quad s_1 = \frac{P}{a}.$$

And if the bending couple $P\Delta$ were acting alone, a resisting moment $\frac{s_2 I}{c}$ would be developed, the unit-stress varying as shown in Fig. 185(b); this resisting moment would hold the external moment $P\Delta$ in equilibrium, and hence

$$P\Delta = \frac{s_2 I}{c}, \quad \text{or} \quad s_2 = \frac{P\Delta c}{I}.$$

The actual unit-stress, then, at any point of the area, would be the algebraic sum of the unit-stresses due to P_1 and to $P\Delta$. Hence the maximum unit-stress s is

$$s = s_1 + s_2$$

$$= \frac{P}{a} + \frac{P\Delta c}{I}, \qquad \qquad (146)$$

and since $I = ar^2$ where r is the radius of gyration, the above equation may be written

$$s = \frac{P}{a}\left(1 + \frac{\Delta c}{r^2}\right). \qquad \qquad (147)$$

Now Δ is eliminated from eq. (147) by assuming that it bears the same relation to the stress in the column as exists between the deflection and stress in beams.[7] This relation for beams is $\Delta = k\frac{sl^2}{c}$ in which k is a constant depending on the kind of material

[7] This assumption does much to destroy the rational basis of the formula since in a beam the stress is due to bending alone, and hence even if the column were an ideal one, only that part of the stress $\left(s - \frac{P}{a}\right)$ which is due to the bending moment $P\Delta$ should be considered to be proportional to $\frac{l^2}{c}$ instead of the total unit-stress s. Further, in order to include the effects of deviations from ideal conditions part of the value of Δ should be considered to be due to initial crookedness.

of which the beams are made, the type of loading, and the end-conditions. And this may be written $\Delta=\phi\frac{l^2}{c}$ where ϕ is a constant similar to k, provided that the beams are subjected to the same maximum unit-stress and that this stress does not exceed the proportional limit of the material. Therefore Δ in eq. (147) is replaced by $\phi\frac{l^2}{c}$; the resulting equation is

$$s=\frac{P}{a}\left(1+\phi\frac{l^2}{r^2}\right) \quad \text{or} \quad \frac{P}{a}=\frac{s}{1+\phi\left(\frac{l}{r}\right)^2}. \qquad (148)$$

This equation is called the Gordon-Rankine column formula; Gordon used the least lateral dimension of the cross-section and Rankine introduced the radius of gyration.

A value of ϕ for columns of a given material and type of end connection is found from results of tests [8] of such columns, the value of ϕ being selected so that the value of P in the equation agrees as well as may be with the average of the experimental values of the loads that caused the columns to fail.

Values of ϕ thus found from experimental data vary widely; the values given in Table 3 are average values that have been used in engineering practice. The value for ϕ recommended by Rankine for wrought-iron pin-ended columns (structural steel was not then manufactured) was $\frac{4}{36000}$ and this value is still frequently used in specifications for structural steel columns, as for example, in the Building Ordnances of many cities. The straight-line formula (Art. 99), however, is now used extensively in the design of structures and also, but to a less extent, in the design of machines.

To determine the maximum load P that a column can resist, the value of s in equation (148) is made equal to the ultimate compressive strength of the material (for ductile material the yield-point should be regarded as the ultimate compressive strength),

[8] In such tests the columns are subjected to approximately the same maximum unit-stress at failure but the value of this stress is greater than the proportional limit of the material, although for ductile material, such as structural steel, the stress in the material when the column fails probably does not exceed the yield-point and hence is not much greater than the proportional limit. However, the method of obtaining ϕ, and the assumption discussed in footnote 7 render the formula empirical rather than rational.

TABLE 3

VALUES * OF ϕ IN RANKINE'S FORMULA

(The values given for "Both Ends pivoted" should be used for all three types of end conditions except for columns that approach closely to an ideal column (as may be the case with columns having solid symmetrical sections) and for slender columns; see note below.)

Material	Both Ends Pivoted	One End Fixed Other End Pivoted	Both ends Fixed
Timber	$\frac{4}{3000}$	$\frac{2}{3000}$	$\frac{1}{3000}$
Cast Iron	$\frac{4}{5000}$	$\frac{2}{5000}$	$\frac{1}{5000}$
Wrought Iron	$\frac{4}{36000}$	$\frac{2}{36000}$	$\frac{1}{36000}$
Structural Steel	$\frac{4}{25000}$	$\frac{2}{25000}$	$\frac{1}{25000}$

* It will be noted that the values of ϕ in this table are in the ratios of 4 : 2 : 1. These ratios were originally selected so as to make them agree with the effect of end conditions in very slender columns as found from Euler's formula (see Art. 100). The columns used in most structures and machines, however, are not slender and in such columns experiments and experience have shown that deviations from ideal conditions such as crookedness in the column as a whole, local kinks, initial stresses, variation in the properties of the material, etc., are the factors that usually control the strength of the column; and that the end-conditions frequently have little effect, except in the case of slender columns and, possibly, columns having solid symmetrical sections. The value of ϕ for columns with intermediate values of l/r as used in most structures and machines, should, therefore, be approximately the same for all three conditions of ends (see also Art. 101), and equal to, or slightly less than, that given for "both ends pivoted."

and to determine a working load the value of s is a working compressive unit-stress for the material. A working value for s ranging from 12,500 to 16,250 lb. per sq. in. is frequently specified for structural steel columns (see footnote 17 and Art. 139) depending somewhat on the value used for ϕ. For example, the Cambria Steel Handbook uses the formula

$$\frac{P}{a}=\frac{12{,}500}{1+\frac{1}{36{,}000}\left(\frac{l}{r}\right)^2}, \quad \ldots \quad (148a)$$

and the Philadelphia Building Laws specify

$$\frac{P}{a}=\frac{16{,}250}{1+\frac{1}{11{,}000}\left(\frac{l}{r}\right)^2}. \quad \ldots \quad (148b)$$

Ritter's Rational Constant.—Ritter proposed that a value of ϕ be derived from the constants of the material instead of from test data. Thus, since the value of $\frac{P}{a}$ from Euler's equation should be equal, when $\frac{l}{r}$ is large, to that from Rankine's equation, we have,

$$\frac{P}{a}=\frac{\pi^2E}{\left(\frac{l}{r}\right)^2}=\frac{s_e}{1+\phi\left(\frac{l}{r}\right)^2}; \quad . \quad . \quad . \quad . \quad . \quad (149)$$

the value of s_e being the proportional limit of the material since a slender column, due to the large deflection, is on the point of failing when, or even before, the proportional limit is reached (see Fig. 182). Now since unity is small compared with $\phi\left(\frac{l}{r}\right)^2$ it may be neglected and hence

$$\phi=\frac{s_e}{\pi^2E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150)$$

which is Ritter's rational constant for use in Rankine's formula for pivot-ended columns. This value of ϕ has been used to some extent in machine design.

ILLUSTRATIVE PROBLEM

Problem 178. The steel parallel rod AB (Fig. 186) of a locomotive has a rectangular cross-section 1 in. by 3 in. and is 6 ft. long. When the locomotive is starting the rod acts mainly as an axially loaded column the cross-bending due to the weight and to the centrifugal forces being negligible. The rod may be considered to be a pin-ended column as regards bending in the vertical plane and a fixed-ended column as regards bending in a horizontal plane; due to the shape of cross-section, the bar tends to bend in the horizontal plane, but the restraint of the pins tends to cause it to bend in a vertical plane as a pin-ended column. Find the working load to which the rod should be subjected, using a working stress equal to one-fourth of the yield-point.

Solution.—The yield-point will be assumed to be 40,000 lb. per sq. in. and hence the working value of s is 10,000 lb. per sq. in.

(a) Treating the rod as a pin-ended column:

$$r^2=\frac{I_x}{a}=\frac{\frac{1}{12}bd^3}{bd}=\tfrac{1}{12}d^2=\tfrac{1}{12}(3)^2=0.75 \text{ in.}^2,$$

whence,

$$\left(\frac{l}{r}\right)^2=\frac{(72)^2}{0.75}=6910 \quad \text{and} \quad \frac{l}{r}=83.2,$$

$$P=\frac{as}{1+\phi\left(\frac{l}{r}\right)^2}=\frac{3\times 10{,}000}{1+\frac{4}{25{,}000}(83.2)^2}=\frac{30{,}000}{1+1.106}$$

$$=14{,}250 \text{ lb., working load.}$$

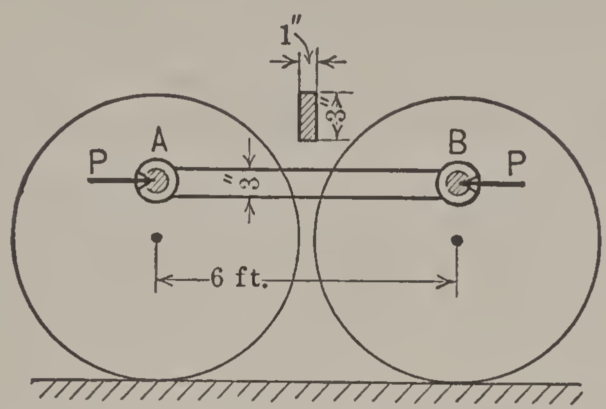

FIG. 186.—Parallel-rod in compression.

(*b*) Treating the rod as a fixed-ended column:

$$r^2=\frac{I_y}{a}=\frac{\frac{1}{12}db^3}{db}=\tfrac{1}{12}b^3=\tfrac{1}{12}in.^2,$$

whence,

$$\left(\frac{l}{r}\right)^2=\frac{(72)^2}{\frac{1}{12}}=62{,}300 \quad \text{and} \quad \frac{l}{r}=249,$$

$$P=\frac{30{,}000}{1+\frac{1}{25{,}000}(249)^2}=\frac{30{,}000}{1+2.48}$$

$$=8630 \text{ lb. working load,}$$

and hence the maximum working load the column should resist is 8630 lb.

PROBLEMS

179. A low-carbon steel bar 8 ft. long and 2.4 in. in diameter is to be tested as a pivot-ended column, the load being applied axially. The yield-point of the material is found to be 36,000 lb. per sq. in. Calculate the slenderness ratio and the maximum load the column would be expected to resist.

Ans. $\frac{l}{r}=160$, $P=32{,}000$ lb.

180. Two 8-in. 25.5-lb. steel I-beams are latticed together, the distance between them being such that the moments of inertia of the cross-section about the two axes of symmetry are equal. The column is pin-ended and is

20 ft. long. Calculate the value of $\frac{l}{r}$ and the maximum unit-stress developed by an axial load of 40 tons.

Ans. $\frac{l}{r}=79.5$; $s=10{,}700$ lb. per sq. in.

181. What should be the spacing of 4×4 in. timber posts 6 ft. long, that support a horizontal platform which is loaded with a uniform load of 200 lb. per sq. ft.? Assume the platform to have no lateral support and hence assume the columns to be fixed at the lower end and free at the upper end; the value of ϕ for such a column will be assumed to be four times that for a pivot-ended column (see Art. 100). Assume a working stress of 1000 lb. per sq. in.

182. The steel connecting-rod on a certain engine is 3 in. by $1\frac{1}{4}$ in. in cross-section and 2 ft. long. It may be assumed to be a pin-ended column for bending about an axis perpendicular to the 3 in. side and a fixed-ended column for bending about an axis parallel to the 3-in. side. What is the greatest pressure that may be applied on the 20-in. piston without exceeding a unit-stress in the rod of 16,000 lb. per sq. in.? Assume the connecting rod to be in its horizontal position.

Ans. 170 lb. per sq. in.

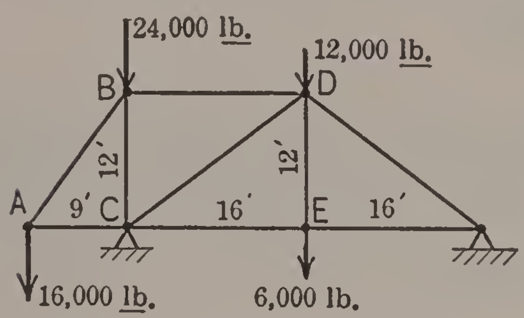

FIG. 187.—Truss containing compression members.

183. The member BC of the pin-connected truss shown in Fig. 187 is made of two 6-in. 15.5-lb. channels latticed together; the moments of inertia about the two principal axes are equal. Calculate the maximum unit-stress in the member.

Ans. $s=7470$ lb. per sq. in.

184. Find the maximum unit-stress in member AC of Fig. 187, assuming it to have the same cross-section as that of member BC described in the preceding problem.

185. A round timber column is 12 in. in diameter and 20 ft. long. If the column has fixed ends, what working axial load may be applied to the column assuming a working stress of 800 lb. per sq. in.

186. A hollow cast-iron column has an outside diameter of 8 in., and inside diameter of 6 in. and a length of 15 ft. If the ends are fixed what is the maximum unit-stress in the column when it carries an axial load of 80,000 lb.?

NOTE.—The problems after Art. 99 may be solved by Rankine's formula if additional problems are desired.

99. Straight-line Formula.—If a series of columns, all made of the same material, with the same general shape of cross-section and the same conditions of ends but with different slenderness ratios, are tested to failure by subjecting them to the same type of loading, the test results will show that, *for columns having intermediate slenderness ratios,* the load per unit area $\left(\frac{P}{a}\right)$ decreases approximately in the same proportion that the slenderness ratio increases. In other words the relation between $\frac{P}{a}$ causing failure and $\frac{l}{r}$ is represented approximately by a straight line.

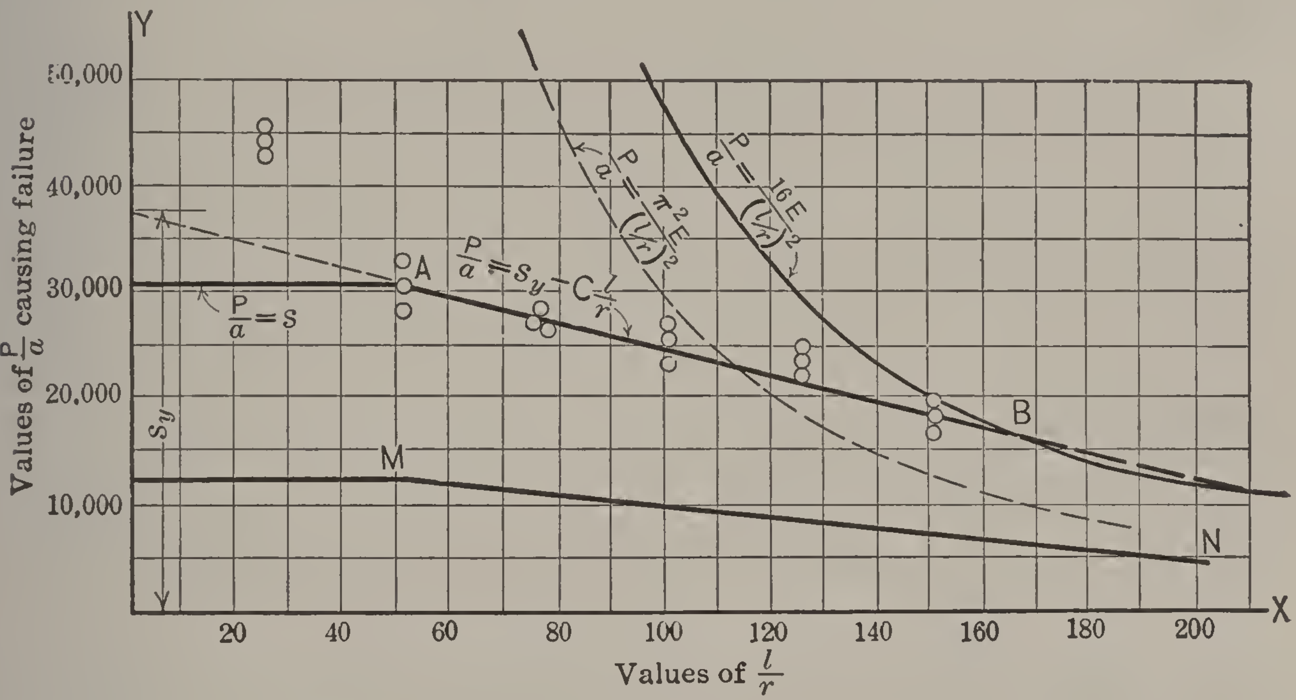

FIG. 188.—Graphical representation of straight-line column formula.

For example, Fig. 188 shows the experimental results obtained from tests [9] of pin-ended axially-loaded structural-steel columns having H-sections. The ordinates represent loads causing failure divided by the cross-sectional area $\left(\frac{P}{a}\right)$ and the abscissas represent the slenderness ratios of the columns $\left(\frac{l}{r}\right)$. The variation [10] in the results for the three columns in each group indicates the effects of the factors (accidental eccentricities, initial crookedness, initial stresses, etc.) discussed in Art. 92. As shown in Fig. 188, for values of $\frac{l}{r}$ between 50 and 150 approximately, a sloping straight line fits the test results fairly well; for values of $\frac{l}{r}$ less than

[9] Made at the Watertown Arsenal; see "Tests of Metals," 1909.

[10] The variations in some tests are much greater than those shown in Fig. 188, particularly in test of large built-up sections.

about 50 the load per unit-area causing failure is considered to be constant and equal approximately to the yield-point of the material, even though laboratory tests of columns *having solid section* sometimes show (as in Fig. 188) that the actual $\frac{P}{a}$ causing failure is greater than the yield-point of the material; tests of built-up columns, however, rarely give values of $\frac{P}{a}$ greater than the yield-point.

Further, the values of $\frac{P}{a}$ found from Euler's formula, $\frac{P}{a}=\frac{\pi^2 E}{\left(\frac{l}{r}\right)^2}$, for ideal pivot-ended column are less[11] than the values of $\frac{P}{a}$ found from tests, when $\frac{l}{r}$ is relatively large (170 to 200). T. H. Johnson[12] found from a careful study of the results of tests of slender columns that if π^2 in Euler's equation were replaced by 16 the resulting equation, $\frac{P}{a}=\frac{16E}{\left(\frac{l}{r}\right)^2}$, would agree well with the test results, for pin-ended slender columns. The curve representing this equation is shown in Fig. 188. Thus, the straight line AB, in Fig. 188 represents the test results only between certain limiting values of $\frac{l}{r}$.

Now the equation of a straight line (similar to AB, Fig. 188) is

$$y=Cx+s_y,$$

in which C is the slope of the line, being negative in Fig. 188, and s_y is the intercept on the y-axis. Therefore the equation of the line AB (Fig. 188) is

$$\frac{P}{a}=s_y-C\frac{l}{r}, \qquad (151)$$

[11] This is probably due to the fact that a pin-ended column does not act as an ideal pivot-ended column; the friction of the pin exerts an end-moment which causes the column to act with an end restraint intermediate between a pivot-ended and a fixed-ended column; the influence of this end restraint being relatively large in the relatively slender columns.

[12] See Transactions of American Society of Civil Engineers, Vol. 15, 1886, p. 517.

which is called the straight-line column formula; it was first proposed by Mr. J. H. Johnson [13] in 1886.

But as already noted, this equation never reduces to $\frac{P}{a}=s_y$ since the value of $\frac{l}{r}$ in the equation cannot be less than that corresponding to the point A (Fig. 188).

The above equation for the straight line [14] shown in Fig. 188 is

$$\frac{P}{a}=37{,}500-125\frac{l}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

and this equation agrees reasonably well with most of the available test data [15] obtained from tests on structural-steel pin-ended, axially-loaded columns in which the full strengths of the columns were developed; that is, the columns did not fail due to local wrinkling or to weak end connections, etc.

Since the value of $\frac{P}{a}$ given by equation (152) is the load per unit-area that causes failure, a working value of $\frac{P}{a}$ may be found by dividing the right-hand member of the equation by a factor (to make allowance for uncertainty of loads, etc., see Art. 7)—a so-called factor of safety. A value of 2.5 is recommended [16] for

[13] Transactions of American Society of Civil Engineers. Vol. 15, 1886, p. 517.

[14] Prof. J. B. Johnson found that a parabola fitted the tests results even better than a straight line, and the following equation of a parabola was found by the Column Committee of the A. R. E. A. to agree well with test results for structural-steel pin-ended columns

$$\frac{P}{a}=32{,}500-\frac{5}{8}\left(\frac{l}{r}\right)^2.$$

By applying a reduction factor of 2.5 the working or design formula becomes

$$\frac{P}{a}=13{,}000-\frac{1}{4}\left(\frac{l}{r}\right)^2.$$

[15] See Report of Sub-Committee on Iron and Steel Structures of the American Railway Engineering Association, Vol. 21, Jan., 1920. Appendix B.

[16] Since this value is based on the yield-point it is equivalent to a value of about 5 based on the ultimate strength; the value commonly used for tension members and beams is 2 based on the yield-point and hence about 4 based on the ultimate strength. For a discussion of this matter see Art. 139.

structural steel columns by a joint committee of the American Railway Engineering Association (A. R. E. A.) and the American Society of Civil Engineers (A. S. C. E.); the working load per unit-area then, is

$$\frac{P}{a}=15{,}000-50\frac{l}{r}, \qquad (153)$$

with a specified *maximum* allowable value for $\frac{P}{a}$ of 12,500 lb. per sq. in. This equation is represented by the line MN in Fig. 188. Thus, the design formulas [17] for structural-steel pin-ended [18] axially-loaded columns so designed that they do not fail due to local wrinkling or to weak end-connections are:

$$\frac{P}{a}=12{,}500, \text{ if } \frac{l}{r} \text{ is not greater than } 50, \qquad (154)$$

$$\frac{P}{a}=15{,}000-50\frac{l}{r}, \text{ if } \frac{l}{r} \text{ lies between 50 and 150} \qquad (155)$$

$$\frac{P}{a}=\frac{16E}{\left(\frac{l}{r}\right)^2} \text{ if } \frac{l}{r} \text{ is greater than 150.} \qquad (156)$$

However, the values of $\frac{P}{a}$ found from the last two of the above equations differ but little for values of $\frac{l}{r}$ between 150 and 210, as

[17] The following formulas were formerly given in the specifications of the A. R. E. A. and are still widely used for the design of columns in buildings, etc.

$$\frac{P}{a}=14{,}000 \text{ if } \frac{l}{r} \text{ is less than } 50,$$

$$\frac{P}{a}=16{,}000-70\frac{l}{r} \text{ if } \frac{l}{r} \text{ is between 50 and 120.}$$

Thus the later specifications changes the former by reducing the maximum permissible unit-stress and by deducting less for the effect of length. It is important to note that the reduction in the permissible stress in columns has been accompanied by a tendency to increase the permissible stresses in beams and tension members, and this is consistent with the discussion in Arts. 91 and 92.

[18] The effect of end conditions is discussed in Art. 100; but specifications usually make no allowance for the end restraint that occurs due to riveted end-connections as used in buildings, bridges, cranes, etc.

indicated in Fig. 188, and hence the second of the above equations may be considered fairly reliable for values of $\frac{l}{r}$ between 50 and 210.

In designing a column, by the straight-line formula, to resist a given load, the dimensions of the cross-section may easily be found directly from the formula, provided that the area is circular or square, but the dimensions of sections of columns made of rolled shapes it is convenient to use the method of trial and error, since the calculations involve quadratic equations. Steel companies' hand books give safe loads on columns having various sections as calculated by different formulas, and these values may be used to secure the first trial value of the section when employing a formula different from that used in the handbook.

Timber and Cast-iron Columns.—The following straight-line formulas giving working values of $\frac{P}{a}$ are representative of those in common use.

$$\frac{P}{a}=1000-6\frac{l}{r}, \text{ for structural timber,} \quad . \quad . \quad . \quad (157)$$

$$\frac{P}{a}=9000-40\frac{l}{r}, \text{ for cast iron.} \quad . \quad . \quad . \quad . \quad . \quad (158)$$

ILLUSTRATIVE PROBLEM

Problem 187. A latticed, pin-ended, steel column is made of two 10-in. 15-lb. channels (Fig. 189). The distance, h, between backs is such that the moments of inertia about the x and y axes are equal. The column is 20 ft. long. Calculate the slenderness ratio and the working axial load for the column.

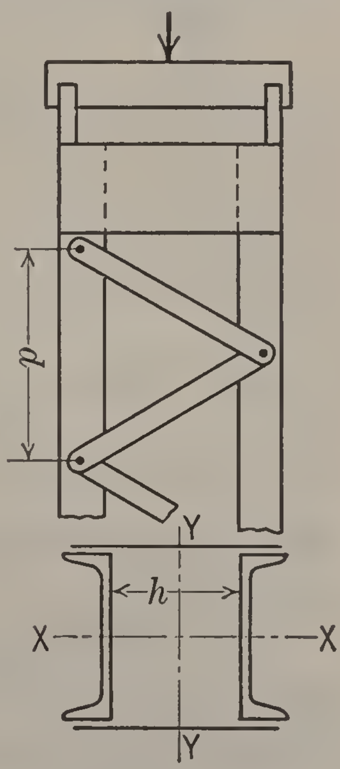

FIG. 189.—Latticed column.

Solution. The following values are found in a steel maker's handbook:

$a=2\times4.46=8.92$ sq. in., and $r=3.87$ in.

$\frac{l}{r}=\frac{240}{3.87}=62.0$, and hence equation 155 is applicable.

$$\frac{P}{a}=15{,}000-50\frac{l}{r}$$

$$=(15{,}000-50\times62)=11{,}900 \text{ lb. per sq. in.}$$

$P=8.92\times11{,}900=106{,}000$ lb., working load.

PROBLEMS

188. A standard 12-in., 40-lb. I-beam 8 ft. long is used as a column with riveted end-connections that are assumed to be equivalent to pin-ends. What working axial load should it carry? (From Cambria handbook; $I_1 = 245.9$ in.^{4}, $I_2 = 10.95$ in.^{4}, $a = 11.76$ in.^{2}. *Ans.* 117,600 lb.

189. A square timber column 16 ft. long resists a working axial load of 120,000 lb. Find the dimensions of the cross-section.
Ans. 13.1 in. square.

190. A column in a certain power-house has an unsupported length of 22 ft. and is made up of a 12 in. by $\frac{1}{2}$ in. plate, to which are riveted four 5 in. by $3\frac{1}{2}$ in. by $\frac{1}{2}$-in. angles as shown in Fig. 190. Find the working axial load.

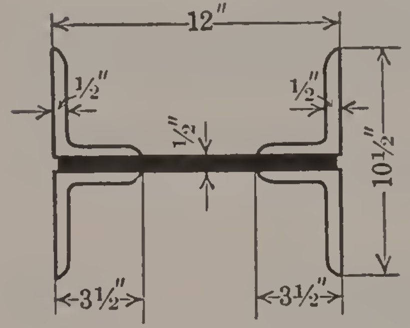

FIG. 190.

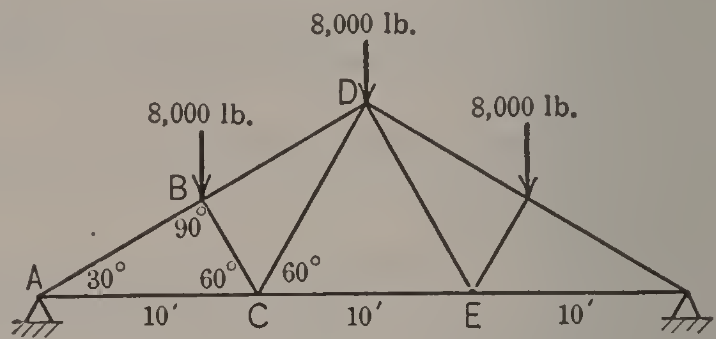

FIG. 191.—Truss containing compression members.

191. What size of angle should be used for the member BC of the truss shown in Fig. 191 (see also Problem 31 and Fig. 33).

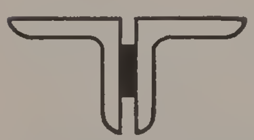

FIG. 192.

192. The upper chord member of a truss (similar to AB or BD in Fig. 191) consists of two angles, separated $\frac{3}{8}$ in. by washers (Fig. 192). If the unsupported length of the chord is 5 ft. and the total compressive axial stress is 65,000 lb. find the size of angles required.
Ans. 5-in. by $3\frac{1}{2}$-in. by $\frac{5}{16}$-in. angles.

NOTE: The problems after Art. 98 may be solved by the straight-line formula if additional problems are desired.

100. Effect of End-conditions.—I. ON IDEAL SLENDER COLUMNS.—The more important ideal end-conditions that are approached more or less closely in the end-connections of structural columns are illustrated in Fig. 193, and may be described as follows:

(*a*) Both ends free to turn but not free to move laterally. Fig. 193(*a*); a column subjected to such end-conditions is usually called a pivot-, round-, or hinge-ended column.

(*b*) Both ends fixed so that the tangents to the elastic curve at the ends are parallel to the original axis of the column, Fig. 193(*b*).

(*c*) One end fixed and one end free to turn but not free to move laterally, Fig. 193(*c*).

(*d*) One end fixed and one end free from all restraint, Fig. 193(*d*).

The axial load P that will cause the column shown in Fig. 193(*a*) to bend (considered to be the maximum load a slender column can resist, see Art. 95) is $P=\frac{\pi^2EI}{l^2}$.

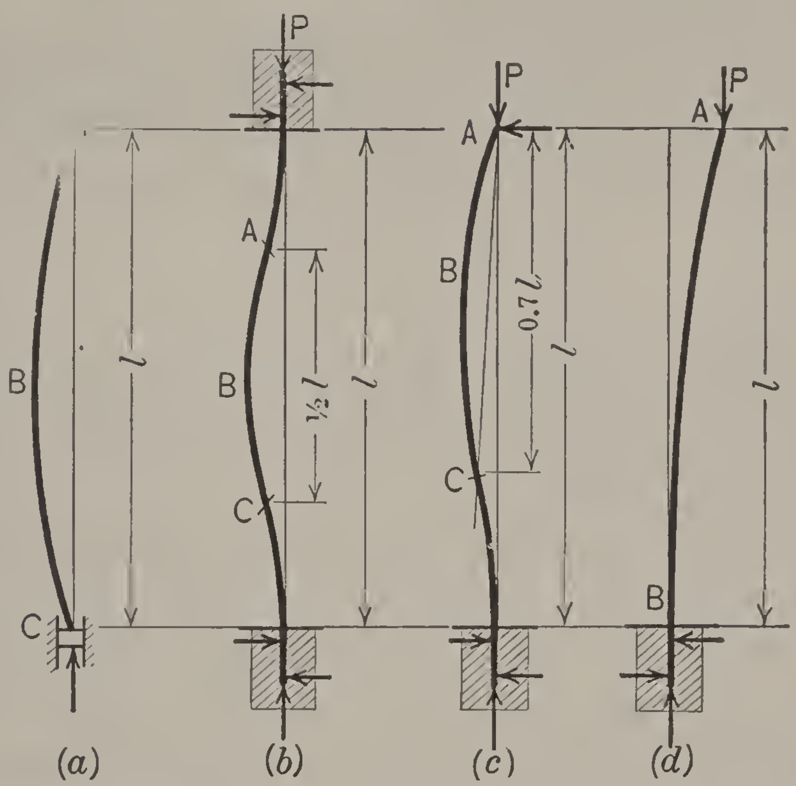

FIG. 193.—Effect of end-conditions on ideal slender columns.

In Fig. 193(*b*) the points of inflection, A and C, are at a distance of $\frac{1}{4}l$ from the ends, and hence the middle half, ABC, is a column of the same type as that in Fig. 193(*a*). Thus the maximum load P for a fixed-ended slender column is

$$P=\frac{\pi^2EI}{\left(\frac{l}{2}\right)^2}=\frac{4\pi^2EI}{l^2}. \qquad (159)$$

Therefore, a fixed-ended slender column of length l will carry as great a load as a pivot-ended column of length $\frac{l}{2}$; that is, a fixed-ended slender column of a given length is four[19] times as strong as a pivot-ended column of the same length.

[19] See footnote 22.

In Fig. 193(*c*) the inflection point C is at a distance of approximately $0.7l$ from A, and the part of the column ABC is of the same type as that of Fig. 193(*a*). Hence,

$$P=\frac{\pi^2EI}{(0.7l)^2}=\frac{2\pi^2EI}{l^2} \text{ approximately.} \qquad (160)$$

Therefore a slender column with one end fixed and the other end pivoted is approximately twice as strong as a column of the same length having both ends pivoted.

In Fig. 193(*d*) the curve AB assumed by the column corresponds to the portion AB of Fig. 193(*a*) and hence Euler's equation for the column in Fig. 193(c) is

$$P=\frac{\pi^2EI}{(2l)^2}=\frac{1}{4}\frac{\pi^2EI}{l^2}. \qquad (161)$$

Therefore, a slender column fixed at one end and free or unrestrained at the other is only one-fourth as strong as a pivot-ended column of the same length.

II. On Columns as Used in Structures and Machines.—The end connections of columns used in structures and machines do not as a rule permit the ideal conditions discussed above to occur. The more common types of structural columns are:

Pin-ended Columns.—A column in a pin-connected structure is usually assumed to approximate closely to a pivot-ended column as regards bending in a plane perpendicular to the axis of the pins, and to approximate more or less closely to a fixed-ended column as regards bending in a plane parallel to the axis of the pins (see discussion under fixed-ended columns below). Tests [20] indicate that the friction of the pin, particularly when the pin is relatively large and the load relatively small, causes the column to act as a fixed-ended column but when subjected to larger loads it acts as a pivot-ended column, and hence in practically all cases the friction should be neglected. Pin-ended columns are commonly used for compression members in bridges, for connecting rods of engines, etc.

Columns with Ends Partially Restrained.—Columns in buildings, bridges, cranes, etc., are frequently fastened to other members by

[20] See tests made by James Christie, Trans. A. S. C. E., 1883, pages 85–122; also "Tests of Large Bridge Columns" by Griffith and Bragg, Technologic Paper No. 101, Bureau of Standards, 1918.

riveted connections but columns with these kinds of end connections cannot have fixed ends since the connections and also the supports to which the column is attached are never rigid. The amount of restraint depends on the relative stiffness of the column end-connections and of the supports to which the column is attached, the stiffer the connections and supports the more the column is restrained at its ends.

As a rule it is impossible to determine in any specific case to what extent the ends are restrained, and in engineering practice the assumption is frequently made that all columns have pivot-ends. However, if the column is known to be well restrained allowance may be made for the increase in strength as indicated in Table 3 and in Art. 102.

Flat-ended or Square-ended Columns.—This type of column bears against approximately plane surfaces of end supports or footings. If the surfaces of the ends of the column and of the footings were true plane surfaces and made perfect contact over the whole end surface of the column, flat-ended columns would approach closely to fixed-ended columns provided that the end supports were relatively rigid. However, there is always much uncertainty as to the way the column bears against the end supports, the unevenness of bearing causing, at working loads, an eccentricity of loading, and hence a so-called flat-ended column is, as a rule, considered to be little if any stronger than a pivot-ended column of the same slenderness ratio. Timber columns are commonly used with so-called flat ends.

101. Rankine's Formula for Columns with Restrained Ends.—In Rankine's formula the effect of end-conditions is introduced in the value of ϕ as indicated in Art. 98 and in Table 3. Now ϕ entered the equation in the term that was assumed to measure the resistance to bending only; further, end restraints affect the resistance to bending mainly, therefore, it has usually been assumed that the values of ϕ for the various types of end restraints (see Table 3) are proportional to the effect of the same end restraints on ideal slender columns as found under I of Art. 100 above, since in slender columns bending is the only action considered; thus, ϕ for fixed ends has usually been assumed to be four times that for pivot-ends, etc., as is shown by the values in Table 3.

But (as already stated after Table 3) the results of tests of columns having intermediate values of slenderness ratios indicate that the type of end conditions may have a much less effect on the strength of the columns than does the deviations from the assumed ideal conditions (see Art. 92), such as crookedness in the column as a whole, local kinks, initial stresses, variations in the properties of the material, etc. For example, some tests of steel columns having slenderness ratios less than 100 show that the strengths of the columns may be influenced but little by any of the end conditions commonly used.

Therefore, except for columns that are relatively slender and for columns that approach closely to ideal columns as may be the case for members with solid symmetrical sections, it is well to assume that the value of ϕ is the same for the different conditions of ends and equal to, or slightly less than, that given for pivot-ended columns (see Table 3 and formulas (148*a*) and (148*b*).

102. Straight-line Formulas for Fixed-ended Columns.—As noted in Art. 100 the strength of an axially loaded slender ideal column with fixed or flat ends is four times that of a similar column having pivot ends. The strength of an ideal column having a very small slenderness ratio, on the other hand, is approximately the same for fixed ends as for pivoted ends since the strength of the column depends only on the strength of the material. Further, since the fixing of the ends increases the resistance of the column to bending chiefly, the influences of fixed ends would be expected to increase with the increase of $\frac{l}{r}$, and test results [21] indicate that this is true.

If test results for $\frac{P}{a}$ and $\frac{l}{r}$ for *fixed-ended* structural-steel columns are plotted, in a way similar to that used in Fig. 188 for pin-ended columns, a straight line is found to fit approximately the average of the test results, the equation of the line being

$$\frac{P}{a}=37{,}500-90\frac{l}{r}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad (162)$$

If a working factor of 2.5 is used, as was done with the equation for pin-ended columns, the resulting equation giving the working load for structural-steel fixed-ended columns is

$$\frac{P}{a}=15{,}000-40\frac{l}{r}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (163)$$

[21] See Bulletin American Railway Engr. Assoc., Vol. 21, Jan., 1920, Appendix B.

in which $\frac{P}{a}$ must not exceed 12,500 lb. per sq. in.; thus for values of $\frac{l}{r}$ less than 62.5 the equation is, $\frac{P}{a}=12{,}500$ lb. per sq. in. The above equations are recommended by the American Railway Engineering Association. The upper limit for $\frac{l}{r}$ to be used in eq. (163) is about 200 and for larger values than 200 Euler's equation[22] is applicable.

Equation (163) should be used only in cases where the restraint of the end connections are comparable to that offered by the relatively rigid heads of a testing machine when making a good contact with the end surfaces of the column.

Buckling of Flange of Beam.—The compression flange of an I-beam or built-up girder is similar to a column with ends partially restrained, and hence may fail by deflecting sidewise. It is customary, therefore, to restrict the maximum compressive unit-stress in the angles or angles and plates which compose the compression flange of a built-up girder; the maximum allowable unit-stress, is stated in the A. R. E. A. specifications to be equal to $\frac{P}{a}$ in the formula

$$\frac{P}{a}=14{,}000-200\frac{l}{b}, \quad . \quad . \quad . \quad . \quad . \quad (164)$$

in which l is the distance between lateral supports and b is the width of the flange. This equation becomes

$$\frac{P}{a}=14{,}000-58\frac{l}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (165)$$

if the flange be regarded as rectangular, since then $b^2=12r^2$ and $b=3.46r$.

103. Straight-line Formula for High Carbon and Alloy Steel Columns.—Formulas 154, 155 and 163 apply to columns made of

[22] Mr. T. H. Johnson found from a careful study (Trans. Am. Soc. C. E., Vol. 15, 1886, p. 517) of test data for slender columns having flat ends that approached closely to fixed ends, that Euler's equation was approximately $\frac{P}{a}=\frac{25E}{\left(\frac{l}{r}\right)^2}$ instead of $\frac{P}{a}=\frac{4\pi^2E}{\left(\frac{l}{r}\right)^2}$ as found for *ideal* slender fixed-ended columns.

steel having an ultimate tensile strength of 55,000 to 65,000 lb. per sq. in. and a yield-point of at least one-half of the ultimate strength, since these are the values stated in the specifications of steel for most structural purposes.

Now if steel having a higher strength than this is used, the strengths of columns having small values of $\frac{l}{r}$ will be increased in about the same ratio as the yield-points (or proportional limits) are increased, whereas the strength of the very slender columns will be increased very little, if any, since the strength of such columns depends on the stiffness (modulus of elasticity) of the material (not the strength of the material) and the stiffness of all grades of steel is practically constant.[23]

There are at present not sufficient test data from which to determine the constants in a straight-line formula for columns of high-strength steel but the few test results[24] available indicate that high-strength steel is of little advantage for slender columns and that, in accordance with the above reasoning, the formula for structural steel columns should be modified for use with high-strength steel by increasing the constant C more than the constant s_y, otherwise the value of $\frac{P}{a}$ will be too high for the larger values of $\frac{l}{r}$. In other words, if the line AB (Fig. 194) represents the straight-line formula for ordinary structural steel the line CD would represent the formula for high-strength steel.

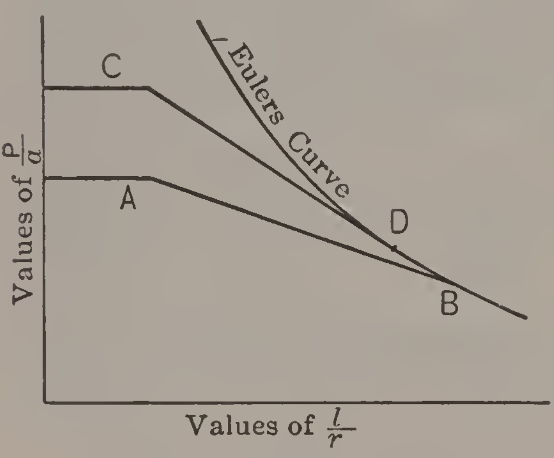

FIG. 194.

For example the safe load for a column made of high-strength

[23] Tubular steel struts sometimes used on airplanes have large values of $\frac{l}{r}$ and hence the struts will be little if any stronger when made of special alloy heat-treated steel than if made of ordinary low-carbon steel, although most of the other steel members of the airplane will be much stronger if made of the high-strength steel.

[24] Bull. Am. Ry. Engr. Assoc., Vol. 21, Jan., 1920, App. B.

steel (having a proportional limit of about 60,000 lb. per sq. in.) as given by the following straight line equation

$$\frac{P}{a}=22{,}500-90\frac{l}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (166)$$

with a maximum value for $\frac{P}{a}$ of 18,750 lb. per sq. in., is comparable with the value of P in the equation $\frac{P}{a}=15{,}000-50\frac{l}{r}$ (with a maximum of 12,500 lb. per sq. in.) for a column made of structural steel, as given in Art. 99.

104. Eccentrically Loaded Columns.—In the preceding articles the column was assumed to be axially loaded. Many columns, however, are subjected to eccentric loads, and the Rankine formula and the straight-line formula may be modified to take account, approximately, of the eccentricity of loading as follows (the modified Euler's equation is discussed in Art. 105):

Modified Rankine's Formula.—In Fig. 195(a) let the column be subjected to a load P having an eccentricity e. This load may be resolved into an axial load P_1 (equal to P) and a couple Pe (Fig. 195). Now the load P_1 if acting alone would cause a unit-stress s_1 on the concave side which, according to Rankine's formula, is

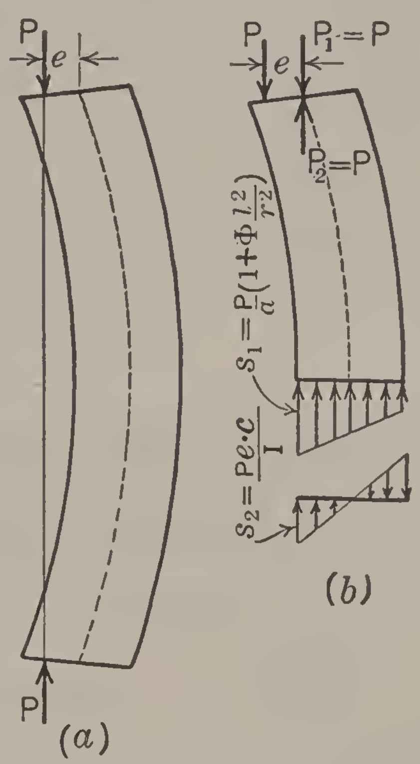

FIG. 195.—Eccentric load on column.

$$s_1=\frac{P}{a}\left(1+\phi\frac{l^2}{r^2}\right),$$

and the bending couple Pe if acting alone would cause, and be held in equilibrium by, a resisting moment $\frac{s_2 I}{c}\left(Pe=\frac{s_2 I}{c}\right.$, see Fig. 195$b$). The unit-stress, s, on the concave side, then, is

$$\begin{aligned} s&=s_1+s_2\\ &=\frac{P}{a}\left(1+\phi\frac{l^2}{r^2}\right)+\frac{Pec}{I}\\ &=\frac{P}{a}\left(1+\phi\frac{l^2}{r^2}+\frac{ec}{r^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (167)\end{aligned}$$

in which r is the radius of gyration of the cross-section about the centroidal axis, and c is the distance from the centroidal axis to the fiber on which the unit-stress s occurs.

Modified Straight-line Formula.—According to the straight-line formula for axial loads (Art. 99) the maximum average unit-stress $\left(\frac{P}{a}\right)$ to which the column may be subjected is

$$\frac{P}{a}=s_y-C\frac{l}{r}. \qquad (168)$$

That is, a column with a given $\frac{l}{r}$ may be considered to be an axially loaded compression block in which the allowable unit-stress is $\left(s_y-C\frac{l}{r}\right)$, provided that the value of $\frac{l}{r}$ is between certain imiting values, as explained in Art. 99.

Now if the load P is applied with an eccentricity e, as in Fig. 195(a), the unit-stress due to the bending moment Pe would be

$$s_2=\frac{Mc}{I}=\frac{Pec}{I}=\frac{Pec}{ar^2},$$

and the maximum unit-stress in the column would be increased by this amount as indicated in Fig. 195(b), and hence the value of $\frac{P}{a}$ must be reduced by the amount s_2.

Thus,

$$\left.\begin{aligned}\frac{P}{a}&=\left(s_y-C\frac{l}{r}\right)-\frac{Pec}{ar^2}\\ \text{or}\qquad \frac{P}{a}\left(1+\frac{ec}{r^2}\right)&=s_y-C\frac{l}{r}.\end{aligned}\right\} \qquad (169)$$

That is, a column with a given $\frac{l}{r}$ and subjected to a load P having an eccentricity e may be treated as an axially loaded short compression block for which the allowable unit-stress is

$$\frac{s_y-C\frac{l}{r}}{1+\frac{ec}{r^2}},$$

provided that the value of $\frac{l}{r}$ is between certain limiting values as explained in Art. 99.

PROBLEMS

193. A timber column supports one end of a beam as indicated in Fig. 196. The column has a square cross-section 12 in. by 12 in. and is 12 ft. long, and is assumed to have ends equivalent to pin ends. If the load is assumed to act with an eccentricity of 4 in., what working load should the column support?

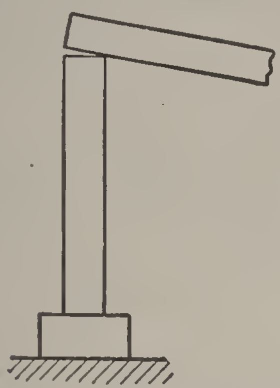
FIG. 196.—Eccentric load on column.

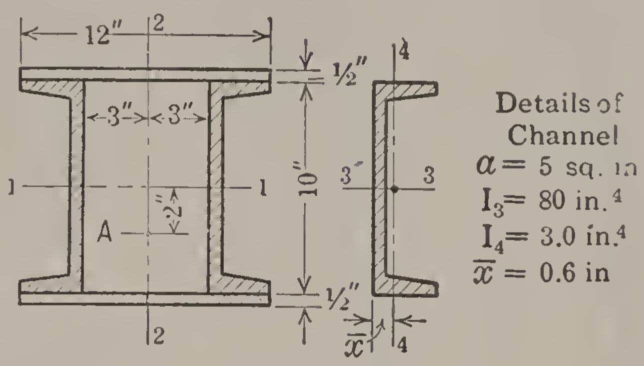

FIG. 197.

194. A pin-ended steel column having the cross-section shown in Fig. 197 is subjected to a load of 40 tons having an eccentricity of 2 in.; the load acts at A, Fig. 197. Calculate the maximum unit-stress in the column.

195. A column is built up of two 10-in., 15-lb. channels laced together (see Fig. 189) so that the distance between backs is 6.33 in. (the moments of inertia about the two principal axes are then equal). The column is 18 ft. long. Calculate the working load if the eccentricity is (*a*) 0 in., (*b*) 2 in., (*c*) 4 in.

105. Equivalent Eccentricity. Secant Formula.—As stated in Art. 97, one method of taking account of the effect of crookedness, lack of homogeneity of material, initial stresses, unintentional or accidental eccentricity, etc., is to assume that these conditions are equivalent, in producing stress, to a positive eccentricity of loading on a straight column of homogeneous material free from initial stresses. Thus, if the eccentricity be denoted by e, the bending moment at any section of the column is $P(e+y)$, as shown in Fig. 198(*a*), instead of Py as was the case in the derivation of Euler's column formula for an axially-loaded column (Art. 95). Now, as shown below, this change in the expression for the bending moment leads to a modification of Euler's formula, called the *secant formula*, from which the maximum unit-stress (and also maximum deflection) caused by a given load and eccentricity can be found; the stress and deflection do not occur in Euler's equation for an axially-loaded column. Further, the secant formula, can be used for eccentrically loaded columns also, since e then is the actual eccentricity plus the equivalent eccentricity.

Secant Formula.—The secant formula may be derived as follows: Let Fig. 198(a) represent a column of length l subjected to a load P having an eccentricity e. Now the equation of the curve assumed by the column is the same as that of a round-ended column of length L (Fig. 198b) subjected to an axial load P, the part MN of the column (Fig. 198b) being in the same state whether the load P is applied at A with eccentricity e or at C with no eccentricity. Now, as stated in Art. 95, when the load P is applied at C as an axial load on a pivot-ended column, the column deflects when the load reaches a value P_1 (say) slightly greater than that of P given by the Euler formula $P=\frac{\pi^2EI}{l^2}$. Further, it was stated in Art. 95 that although there is a definite deflection corresponding to each value of P_1, a very small increase in the load causes a very large increase in the deflection, and hence we may assume that when the column bends, its position may be that of any one of the several curves CD in Fig. 198(b). But, for any given eccentricity e and length l, as indicated in Fig. 198(b), there is only one curve that the column can assume, and the maximum deflection Δ and the maximum unit-stress due to the bending moment $P(e+\Delta)$ do not disappear from the elastic curve equation as was the case in the treatment of the axially-loaded slender column in Art. 95.

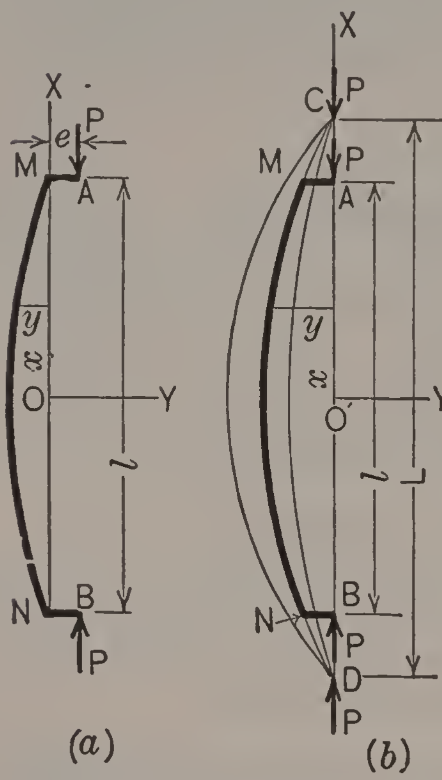

Fig. 198.—Equivalent eccentricity

Now, from Art. 95, the equation of any one of the curves CD (Fig. 198b) is

$$y=\Delta \sin \sqrt{\frac{P}{EI}}x, \quad . \quad . \quad . \quad . \quad . \quad . \quad (170)$$

if the axes are chosen with the origin at C; but if the axes are chosen with the origin at the center of the column length (at O', Fig. 198), the equation is (see footnote for Art. 95).

$$y=\Delta \cos \sqrt{\frac{P}{EI}}x. \qquad (171)$$

But y in this equation is the deflection of the pivot-ended column of length L (Fig. 198b) and hence if y is made to represent the deflection of the eccentrically-loaded column as shown, Fig. 198(a), the equation of the elastic curve assumed by the column is

$$y+e=(\Delta+e) \cos \sqrt{\frac{P}{EI}}x. \qquad (172)$$

Now $y=0$ when $x=\frac{l}{2}$. Therefore,

$$e=(\Delta+e) \cos \frac{l}{2}\sqrt{\frac{P}{EI}}, \qquad (173)$$

and hence the maximum deflection of the eccentrically-loaded column is

$$\Delta=e\left(\sec \frac{l}{2}\sqrt{\frac{P}{EI}}-1\right), \quad \text{or}$$

$$\Delta=e\left(\sec \frac{l}{2r}\sqrt{\frac{P}{aE}}-1\right). \qquad (174)$$

The maximum bending moment is

$$\begin{aligned} M&=P(\Delta+e)\\ &=P\left[e\left(\sec \frac{l}{2r}\sqrt{\frac{P}{aE}}-1\right)+e\right]\\ &=Pe \sec \frac{l}{2r}\sqrt{\frac{P}{aE}}. \qquad (175)\end{aligned}$$

The maximum fiber unit-stress s is

$$\begin{aligned} s&=\frac{P}{a}+\frac{Mc}{I}\\ &=\frac{P}{a}+\frac{Pec \sec \frac{l}{2r}\sqrt{\frac{P}{aE}}}{ar^2}\\ &=\frac{P}{a}\left(1+\frac{ec}{r^2}\sec \frac{l}{2r}\sqrt{\frac{P}{aE}}\right). \qquad (176)\end{aligned}$$

The expression $\frac{l}{2r}\sqrt{\frac{P}{aE}}$ is an angle expressed in radians and is called the *Eulerian angle.* The above equation may also be written in the form

$$P=\frac{as}{1+\frac{ec}{r^2}\sec\frac{l}{2r}\sqrt{\frac{P}{aE}}}. \quad . \quad . \quad . \quad . \quad . \quad (177)$$

The above equations are valid only for values of s within the proportional limit of the material since the elastic curve equation and the flexure formula are used in their derivations. But within the proportional limit they are valid provided that the column acts as a unit and hence does not fail due to local wrinkling or weak end connections. And when these conditions are satisfied the equivalent eccentricity e may be selected so that the calculated values of $\frac{P}{a}$ for axially loaded columns are made to agree as well as may be with experimental results. The secant formula has been presented from time to time in technical literature as being as satisfactory a solution of column action as is likely to be obtained, but due largely to its relative y complicated form and to insufficient experimental data for determining reliable values of the equivalent eccentricity it is not in general use.

When the dimensions (a, c, r, and l) of the column are given and a value of e is known or assumed, the load P required to cause a specified unit-stress, s, may be found from equation (176). And likewise the area a needed to keep the unit-stress from exceeding the prescribed value, s, when the column is subjected to a given load, P, may be found if values of c, r and s are known. But in equation (176) since the quantities P and a are not expressed explicitly in terms of the other quantities, the solution of the equation is most conveniently made by trial and error. To aid in the solution of equation (176) values of $\frac{l}{2r}\sqrt{\frac{P}{aE}}$ for various values of $\frac{P}{a}$ and $\frac{l}{r}$ are given in Table 4 for use in the investigation of *steel* columns ($E=30{,}000{,}000$ lb. per sq. in. for steel).

Values of e and of $\frac{ec}{r^2}$.—The value of the equivalent eccentricity e to be used in equation (176) for axially loaded columns must be assumed, the value depending on the judgment of the engineer. Moncrief,[25] from the analysis of the results of many tests of columns found the value of $\frac{ec}{r^2}$ to range from 0.15 to 0.6, and he recommended the larger value to be on the safe side. He found that this value applies alike to cast iron, wrought iron, mild steel, high carbon steel and several kinds of timber.

TABLE 4

VALUES OF SEC $\frac{l}{2r}\sqrt{\frac{P}{aE}}$ IN SECANT FORMULA

	Values of $\frac{l}{r}$.									
Values of $\frac{P}{a}$	50	60	70	80	90	100	110	120	130	140
5,000	1.05	1.08	1.11	1.15	1.20	1.25	1.32	1.40	1.50	1.62
8,000	1.09	1.14	1.19	1.26	1.35	1.46	1.60	1.79	2.05	2.41
10,000	1.12	1.17	1.25	1.34	1.47	1.63	1.86	2.18	2.65	3.46
14,000	1.17	1.25	1.37	1.54	1.77	2.12	2.68	3.69	6.02	16.4
18,000	1.22	1.35	1.53	1.79	2.21	2.95	4.51	9.86		
20,000	1.25	1.40	1.62	1.95	2.51	3.62	6.65	47.40		
22,000	1.28	1.45	1.71	2.13	2.90	4.65	12.3			
25,000	1.33	1.54	1.88	2.47	3.73	7.87				
28,000	1.38	1.64	2.08	2.93	5.15	22.90				
30,000	1.42	1.72	2.23	3.32	6.79					
32,000	1.46	1.79	2.41	3.83	9.86					
35,000	1.52	1.92	2.73	4.93	28.70					
38,000	1.59	2.07	3.13	6.80						
40,000	1.63	2.18	3.46	9.07						
42,000	1.68	2.31	3.87	13.30						
45,000	1.76	2.51	4.68	43.00						
48,000	1.85	2.76	5.88							
50,000	1.91	2.95	7.06							

Values of *sec* $\frac{l}{2r}\sqrt{\frac{P}{aE}}$

For a circular cross-section, if $\frac{ec}{r^2}=0.6$, $e=0.3r$ and hence for a column having a circular section 10 in. in diameter, e would equal $\frac{3}{4}$ in.

[25] Trans. Am. Soc. Civ. Engrs., Vol. 45, 1901, p. 334.

Prichard [26] proposes the following:

$$\frac{ec}{r^2}=\frac{1}{10}+\frac{1}{700}\cdot\frac{l}{r}. \qquad (178)$$

And when this is applied to ordinary structural-steel pin-ended columns in which the value of $\frac{c}{r}$ is usually about $\frac{7}{5}$,

$$e=0.07r+0.001\frac{l}{r}. \qquad (179)$$

Thus a column having the relatively large $\frac{l}{r}$ of 100 ($r=5$ and $l=500$) the value of e is 0.85 in.

Basquin [27] proposes the following:

$$\frac{ec}{r^2}=0.01+0.001\frac{l}{r}. \qquad (180)$$

In Fig. 199 are shown the curves representing the secant formula

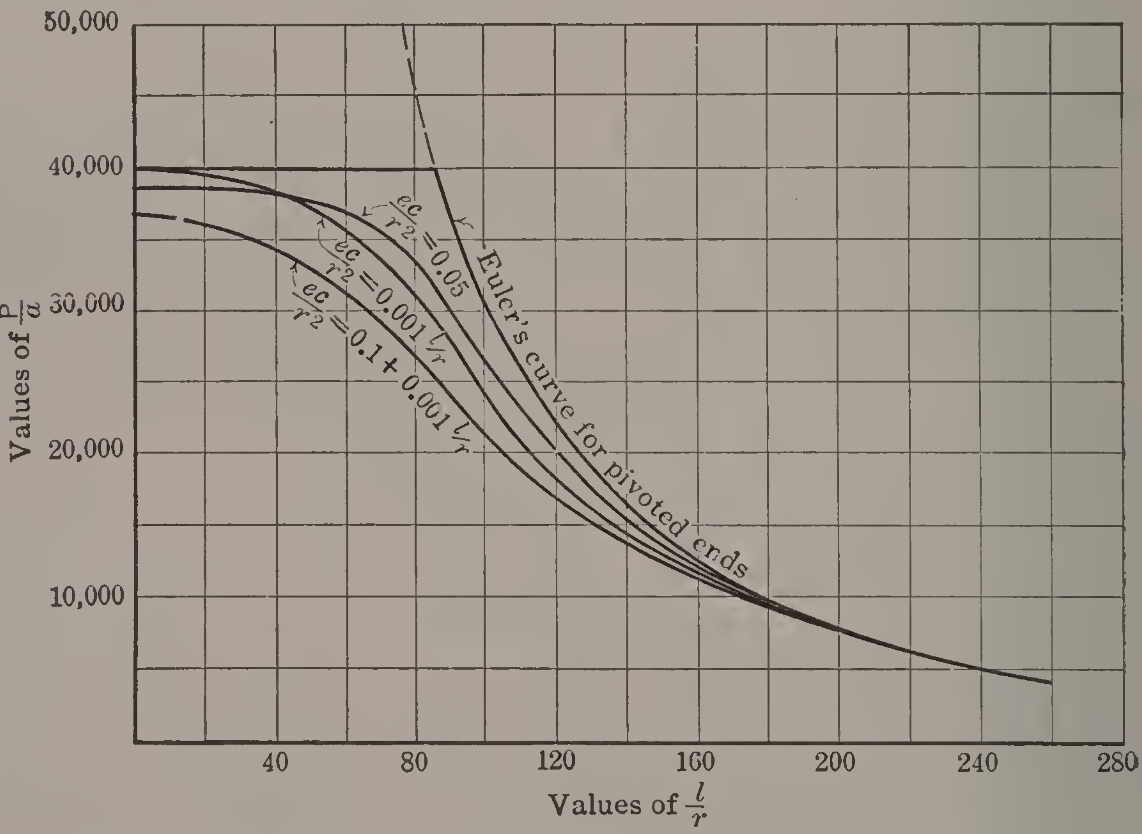

FIG. 199.—Effect of values of $\frac{ec}{r^2}$ on secant column formula.

for steel columns, with different values of the eccentric ratio $\frac{ec}{r^2}$; the yield-point of the steel is assumed to be 40,000 lb. per sq. in.

[26] Trans. Am. Soc. Civ. Engrs., Vol. 61, p. 173.

[27] Journal Western Society Engrs., Vol. 18, 1913, p. 457. For a compilation of the recommendation of many authorities see Salmon's "Columns," Oxford Technical Publications, p. 148. This book also gives an excellent discussion of the factors entering in the various column formulas.

PROBLEMS

196. A column having the same cross-section as the column described in Prob. 172 (see Fig. 183) and a length of 25 ft. is subjected to an axial load of 120,000 lb. Find the maximum unit-stress, assuming that $\frac{ec}{r^2}=0.5$.

Ans. $s=8530$ lb. per sq. in.

197. The column described in Prob. 183 is subjected to an axial load of 73,000 lb. Find the maximum unit-stress in the column; use eq. (178) for finding a value of $\frac{ec}{r^2}$.

Ans. $s=9900$ lb. per sq. in.

198. Solve Prob. 180 by the secant formula using a value of 0.6 for $\frac{ec}{r^2}$.

199 Change the length of the column described in Prob. 192 to 6 ft. and solve by the secant formula using a value for $\frac{ec}{r^2}$ as given by eq. (180).

106. Built-up Steel Columns.—In deriving the column formulas in the preceding articles it was assumed that bending or flexure existed in the column as a whole, the deflection of the axis of the column being the result of initial eccentricity of the load, lack of homogeneity in the material, a general bend or lack of straightness in the column as a whole, or a combination of two or more of these conditions. Now the resistance of the column to bending is increased by giving the cross-sectional area as large a radius of gyration as possible, thereby causing $\frac{l}{r}$ to be small. This is accomplished in built-up columns by using thin component parts (plates, channels, angles, etc.), and placing the parts as far from the neutral axis as practicable, thereby tending to produce a flimsy column. To what extent the cross-section may thus be distended without causing the column to fail locally by wrinkling (secondary flexure) cannot be definitely known but it is probable that some of the disastrous failures of columns have been due to wrinkling [28] or to weak end connections.

In order to help prevent local failure it is the practice in good design of built-up columns that the slenderness ratio of each

[28] For a column formula which takes account of the possible failure by wrinkling, see "Strength of Columns" by W. E. Lilly, Trans. Am. Soc. Civ. Engrs., 1913.

unsupported part of the built-up column shall not exceed the $\frac{l}{r}$ of the whole column. For example, the $\frac{l}{r}$ of the length, p, between lattice bar connections of each channel in Fig. 189 should not exceed the $\frac{l}{r}$ of the whole column. However, this requirement is hardly sufficient since at failure the column is so bent that the maximum fiber unit-stress due to combined compression and bending is equal to the yield-point of the steel, and hence the component part should be designed so that it will not fail below the yield-point, therefore the $\frac{l}{r}$ of the component parts should be limited to about 40, otherwise, if a component part located near the point of maximum bending (near the center) is a relatively slender column (even though it has the same $\frac{l}{r}$ as the whole column) it would fail without developing the yield-point strength of the material.

Further, test results [29] of built-up columns show that the effects of local kinks, initial stresses due to riveting, etc., local eccentricity of loading due to the method of tying the component parts together (interior eccentricity) and variation in strength of component parts may, and probably have, a more important effect in causing local flexure or wrinkling than has the $\frac{l}{r}$ of the part in question. Again, even though these conditions do not cause the column to fail by local wrinkling they may be the cause of the initial bending which leads to the bending of the column as a whole.

It must be remembered also that high localized stress in a column is more serious, in general, in its effect on the strength of a column that it is on the resistance of a tension or flexural member. For these reasons the working or allowable unit-stress for built-up structural steel columns is low compared with that for similar material when used in tension members and beams. For example,

[29] "An Investigation of Built-up Columns under Load," Bulletin No. 44 Engineering Experiment Station, University of Illinois; also, " Tests of Large Bridge Columns " by Griffith and Bragg, Technologic Paper No. 101, Bureau of Standards.

a maximum allowable unit-stress of 12,500 lb. per sq. in. is specified for structural steel columns whereas 16,000 to 18,000 lb. per sq. in. is allowed for tension and flexural members, and this maximum value of 12,500 is reduced as $\frac{l}{r}$ increases according to the formulas of Art. 99, for, although factors other than $\frac{l}{r}$ may have a controlling influence, as discussed above, there is no satisfactory way to measure their effect (unless it is the equivalent eccentricity discussed in Art. 105 which leads to a complicated formula), and a column formula that is made to agree with test results may be used with reasonable confidence even though it may not be rational, provided that the column is designed so that failure will not occur by wrinkling or due to weak end connections, and that the column is similar to those used in obtaining the test results.

107. Columns Subjected to Cross-Bending.—A compression member that is subjected to cross-bending loads may be considered to be (1) a *beam* subjected to end thrust as discussed in Art. 86, or (2) a *column* subjected to cross-bending loads; depending on the relative magnitudes of the end thrust and cross-bending loads, and on the dimensions of the member. In some cases both methods should be used in designing or investigating the member.

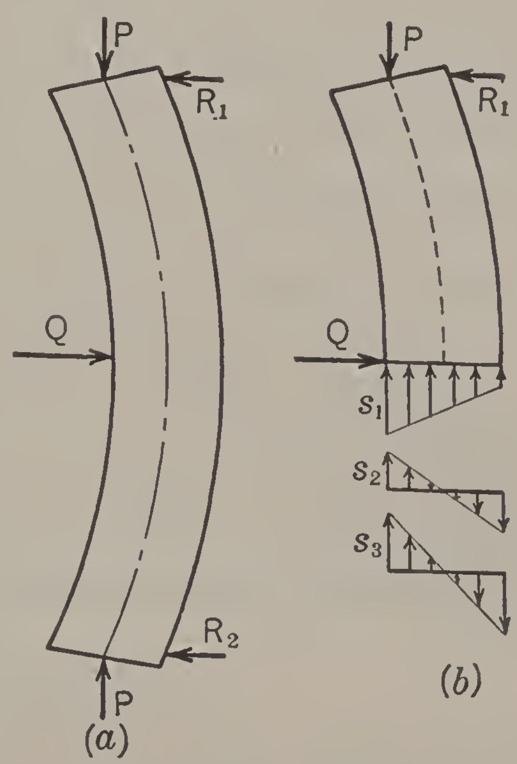

FIG. 200.—Column subjected to transverse loads.

Let Fig. 200 represent a pin-ended column subjected to an axial end load P and a cross-bending load Q. An approximate value of the unit-stress developed may be found as follows:

Modified Rankine's Formula.—According to Rankine's formula the load P if acting alone would produce a maximum unit-stress s_1 on the concave side (Fig. 200b) such that

$$s_1 = \frac{P}{a}\left(1 + \phi\frac{l^2}{r^2}\right).$$

Now if the deflection of the member as a beam, due to Q alone, be denoted by Δ (Δ in this case equals $\frac{1}{48}\frac{Ql^3}{EI}$, Art. 47) then the additional stress caused by P due to the moment $P\Delta$ is

$$s_2=\frac{P\Delta c}{I}=\frac{P\Delta c}{ar^2}.$$

This is equivalent to assuming that the column is straight and that the load is applied with an eccentricity of Δ. Further, if the moment due to the cross-bending load alone be denoted by M (in this case $M=\frac{1}{4}Ql$, Prob. 87) the unit-stress s_3 caused by this moment is

$$s_3=\frac{Mc}{I}=\frac{Mc}{ar^2}.$$

And if the proportional limit is not exceeded, the unit stress, s, on the concave side is

$$\begin{aligned} s&=s_1+s_2+s_3\\ &=\frac{P}{a}\left(1+\phi\frac{l^2}{r^2}\right)+\frac{P\Delta c}{ar^2}+\frac{Mc}{ar^2}. \qquad (181)\end{aligned}$$

Modified Straight-line Formula.—By use of the straight-line formula Δ may be considered an eccentricity, and equation (169) of Art. 104 may be used, from which the unit-stress due to the cross-bending must be subtracted. Hence

$$\frac{P}{a}=\left(s_y-C\frac{l}{r}\right)-\frac{P\Delta c}{ar^2}-\frac{Mc}{ar^2}. \qquad (182)$$

Modified Secant Formula.—Similarly the secant formula (equation (169) of Art. 105) may be used, the value of e being made equal to Δ plus the assumed equivalent eccentricity e_1, and the unit-stress due to the cross-bending load being added to the unit-stress due to the end load. Thus,

$$s=\frac{P}{a}\left(1+\frac{(e_1+\Delta)c}{r^2}\sec\frac{l}{2r}\sqrt{\frac{P}{aE}}\right)+\frac{Mc}{ar^2}. \qquad (183)$$

The values given by these equations err on the side of danger since the deflection of the column is greater than that assumed, but a more refined analysis, as a rule, is not practicable.

PROBLEM

200. Consider the member BC in Fig. 161 to be a pin-ended column subjected to cross-bending and calculate the maximum unit-stress in the member.

CHAPTER XII

COMBINED NORMAL AND SHEARING STRESSES

(Chapter III should be reviewed before this chapter is studied.

108. Introduction.—The problem discussed in this chapter may be stated as follows: Given, at a point in a body, a shearing unit-stress on each of two planes at right angles to each other, and a normal unit-stress on one of the two planes; find the values of the resulting maximum normal and shearing unit-stresses at the same point, in terms of the given stresses, and the directions of the planes on which these maximum stresses occur.

Many structural and machine members are subjected to loads that cause the "given" stresses stated above, and these stresses may be found from the loads, by use of the equations developed in the preceding chapters. The given stresses may arise in the following ways:

(*a*) The shearing unit-stress, s_s, and the normal unit-stress, s, may be due to central loads (Art. 3), as in a bolt loaded as shown in Fig. 201, in which case s_s and s are found from the loads as follows:

$$s_s=\frac{Q}{a} \quad \text{and} \quad s=\frac{P}{a}. \qquad (184)$$

(*b*) s_s may be due to torsional loads and s to an axial load (Fig. 202), in which case (see Art. 26).

$$s_s=\frac{Tc}{J} \quad \text{and} \quad s=\frac{P}{a}. \qquad (185)$$

(*c*) s_s may be due to torsional loads and s to bending loads, as in an engine crank shaft, etc., in which case (see Art. 26 and 34)

$$s_s=\frac{Tc}{J} \quad \text{and} \quad s=\frac{Mc}{I}. \qquad (186)$$

(*d*) Both s_s and s may occur in a beam in which case (see Art. 40)

$$s_s = \frac{V}{It}\int_{y_0}^{c} y\,da \quad \text{and} \quad s = \frac{Mc}{I}. \qquad (187)$$

Further, as shown in Art. 17, if a shearing unit-stress of intensity s_s occurs in one plane a shearing unit-stress equal to s_s must occur on a plane at right angles to the first.

109. Maximum Normal and Shearing Stresses.—Fig. 201 shows a bolt loaded, as suggested under (*a*) above, so as to produce the combination of stresses under discussion, the stresses being

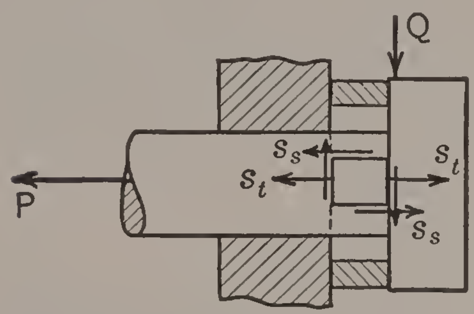

FIG. 201.—Combined shearing tensile stresses.

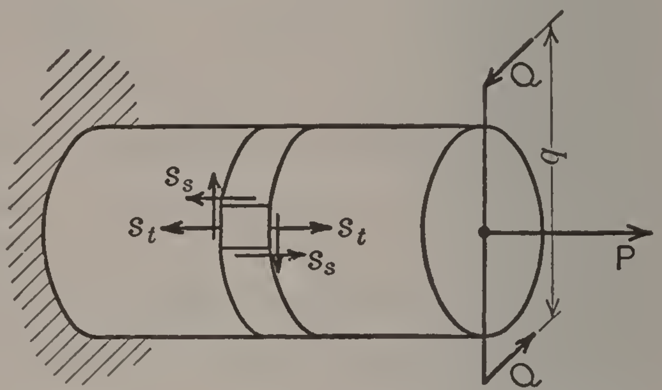

FIG. 202.—Combined shearing and and tensile stresses.

shown on the elementary block in the bolt, and Fig. 202 shows a similar block subjected to the same combination of stresses by the method of loading suggested under (*b*) above. Fig. 203(*a*) shows one of these elementary blocks, enlarged, and also the forces that

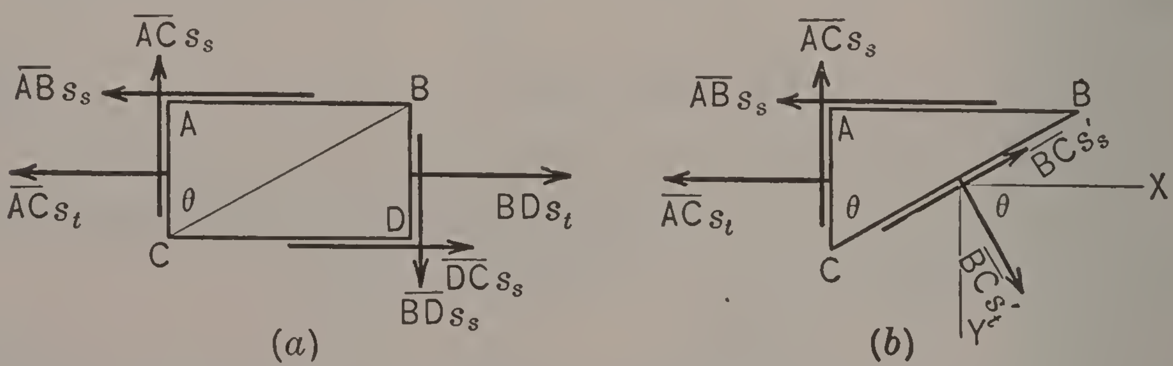

FIG. 203.—Combined shearing and tensile stresses.

act on its four faces and that hold it in equilibrium; the dimension perpendicular to the plane of the paper is assumed for convenience to be unity.

The problem is to find the value of the maximum normal and shearing unit-stresses resulting from this combination of stresses, and the directions of the planes on which these maximum unit-stresses occur.

To Find the Maximum Normal Stress.—Let a plane be passed through the elementary block of Fig. 203(a) at an angle θ with the plane on which the normal stress occurs. The normal and shearing unit-stresses on this inclined plane will be denoted by s'_t and s'_s, respectively. The forces, then, that hold the block ABC in equilibrium are as indicated in Fig. 203(b). And since the sum of the components of the forces in the x-direction and in the y-direction must each be equal to zero, we may write,

$$\overline{BC}s'_t \cos\theta + \overline{BC}s'_s \sin\theta - \overline{AB}s_s - \overline{AC}s_t = 0, \quad . \quad (188)$$

and

$$\overline{BC}s'_t \sin\theta - \overline{BC}s'_s \cos\theta - \overline{AC}s_s = 0. \quad . \quad . \quad (189)$$

By dividing each of these equations by $\overline{BC}$ and noting that $\frac{AB}{BC} = \sin\theta$ and $\frac{AC}{BC} = \cos\theta$ the resulting equations are:

from (188),

$$(s'_t - s_t)\cos\theta + (s'_s - s_s)\sin\theta = 0, \quad . \quad . \quad . \quad (190)$$

from (189)

$$s'_t \sin\theta - (s_s + s'_s)\cos\theta = 0. \quad . \quad . \quad . \quad . \quad . \quad (191)$$

But, *the normal unit-stress, s'_t, has its maximum or minimum value on the plane on which the shearing unit-stress, s'_s, is zero* (see Art. 110 for proof).[1] Therefore, if s'_s in the above equations is made equal to zero, and if θ' is used to denote the value of θ when s'_t is a maximum or minimum, the above equations become,

$$(s'_t - s_t)\cos\theta' - s_s \sin\theta' = 0, \quad . \quad . \quad . \quad . \quad (192)$$

$$s_s \cos\theta' - s'_t \sin\theta' = 0. \quad . \quad . \quad . \quad . \quad (193)$$

By eliminating θ' from these two equations, the maximum and minimum values of s'_t may be found in terms of s_t and s_s which in turn may be found from the external loads by the methods already discussed. And by eliminating s'_t from the two equations the

[1] This theorem was involved in a restricted form in Art. 35 where it was proved that the maximum fiber (normal) stress occurs on the section for which the shear is zero. It is also illustrated in Art. 16 where s_n is shown to be always less than s, the stress on the plane where there is zero shearing stress.

values of θ' may be found which give the directions of the planes on which the maximum and minimum normal stresses occur.

By dividing (192) by (193) the following equation is obtained:

$$\frac{s'_t - s_t}{s_s} = \frac{s_s}{s'_t},$$

or

$$s'^2_t - s_t s'_t - s_s^2 = 0.$$

Therefore,

$$s'_t = \tfrac{1}{2}s_t \pm \sqrt{\left(\frac{s_t}{2}\right)^2 + s_s^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (194)$$

or

$$s'_t = \tfrac{1}{2}s_t \pm \tfrac{1}{2}\sqrt{s_t^2 + 4s_s^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (195)$$

Thus, there are two principal [2] unit-stresses, one being the *maximum* normal unit-stress (in this case a tensile stress) given by the expression

$$\text{max. } s'_t = \tfrac{1}{2}s_t + \tfrac{1}{2}\sqrt{s_t^2 + 4s_s^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (196)$$

and the other the *minimum* normal unit-stress given by the expression

$$\text{min. } s'_t = \tfrac{1}{2}s_t - \tfrac{1}{2}\sqrt{s_t + 4s_s^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (197)$$

But the last term of the right side of this equation is always greater than $\frac{1}{2}s_t$, and hence s'_t is a negative tensile stress, that is, a compressive stress, and may be written

$$-s'_t = s'_c = -\tfrac{1}{2}s_t + \tfrac{1}{2}\sqrt{s_t^2 + 4s_s^2}. \quad . \quad . \quad . \quad . \quad (198)$$

Further, the planes on which the maximum and minimum normal unit-stresses occur are at right angles to each other as is proved below.

If now the normal stress s_t in Fig. 203(a) had been a compressive stress, s_c, then the maximum normal stress would have been a compressive stress given by the equation

$$\text{max. } s'_c = \tfrac{1}{2}s_c + \tfrac{1}{2}\sqrt{s_c^2 + 4s_s^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (199)$$

and the minimum normal stress would have been a negative compressive stress, that is, a tensile stress; namely,

$$-s'_c = s'_t = -\tfrac{1}{2}s_c + \tfrac{1}{2}\sqrt{s_c^2 + 4s_s^2}. \quad . \quad . \quad . \quad . \quad (200)$$

[2] A principal stress (see Art. 19) is one that occurs on a plane on which no shearing stress exists, and is always the maximum or minimum normal stress at the point under consideration.

Direction of Planes on Which Principal Stresses Occur.—The value of θ' in equations (192) and (193) may be found by eliminating s'_t from these equations:

From 193,

$$s'_t = s_s \frac{\cos\theta'}{\sin\theta'}.$$

Substituting this value of s'_t in (192) we have

$$s_s \frac{\cos^2\theta'}{\sin\theta'} - s_t \cos\theta' = s_s \sin\theta',$$

or,

$$s_s (\cos^2\theta - \sin^2\theta) = s_t \sin\theta' \cos\theta',$$

whence,

$$s_s \cos 2\theta' = \tfrac{1}{2} s_t \sin 2\theta'.$$

Therefore,

$$\tan 2\theta' = \frac{2s_s}{s_t}. \qquad \qquad (201)$$

Now there are always two angles, between 0° and 360°, for which the tangents are equal, the two angles differing by 180°. Thus, there are two values of $2\theta'$ that differ by 180°, and hence there are two values of θ' that differ by 90°, as indicated in Fig. 204. Therefore, *the principal stresses occur on planes that are perpendicular to each other*, and the direction of the principal planes may be found by solving equation (201) for θ', θ' being measured from the plane on which the normal stress, s_t, occurs. The position of an elementary block on which the principal stresses act, as they occur in the bolt of Fig. 201, is shown in Fig. 205(a).

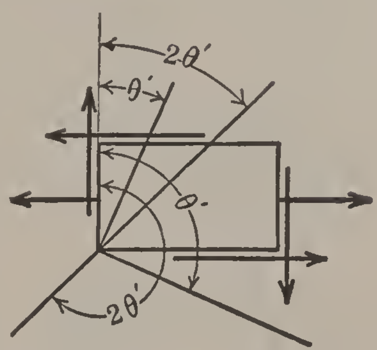

FIG. 204.—Planes on which principal stresses act.

The Maximum Value of the Shearing Unit-stress s'_s.—As shown in Art. 20, the maximum shearing unit-stress resulting from two principal stresses is one-half the algebraic difference of the principal stresses. Therefore,

$$\text{max. } s'_s = \tfrac{1}{2}(\text{max. } s'_t - \text{min. } s'_t) = \tfrac{1}{2}(s'_t + s'_c)$$

$$= \tfrac{1}{2}\left(\tfrac{1}{2}s_t + \tfrac{1}{2}\sqrt{s_t^2 + 4s_s^2}\right) + \left(-\tfrac{1}{2}s_t + \tfrac{1}{2}\sqrt{s_t^2 + 4s_s^2}\right).$$

Hence,

$$\left.\begin{aligned} &\text{max. } s'_s = \tfrac{1}{2}\sqrt{s_t^2 + 4s_s^2}, \\ \text{or} \qquad &\text{max. } s'_s = \sqrt{\left(\frac{s_t}{2}\right)^2 + s_s^2}. \end{aligned}\right\} \quad . \quad . \quad (202)$$

And, as shown in Art. 20, this maximum shearing unit-stress occurs on each of two planes that make angles of 45° with each of the planes on which the principal stresses occur. That is, *the planes of maximum shear bisect the angles between the planes on which the principal stresses occur.* The planes on which the maximum shearing stresses occur are shown in Fig. 205(*b*) in which $\theta'_s = \theta' + 45°$. There are normal stresses also on the planes on which the maximum shearing stresses occur but these are not shown in Fig. 205(*b*).

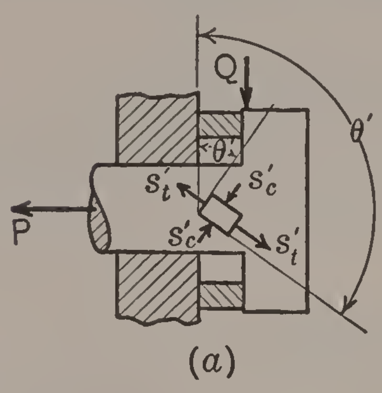

FIG. 205.—Principal stresses and maximum shearing stresses in bolt.

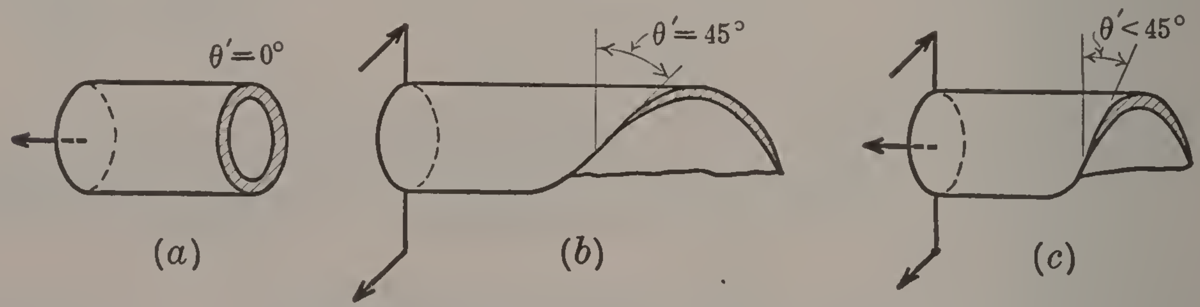

FIG. 206.—Directions of principal planes shown by areas of rupture of brittle material.

The student should draw diagrams similar to Fig. 205 in connection with a shaft when loaded as indicated in Fig. 202, and explain why the cast-iron specimens shown in Fig. 206 failed as indicated. (See also Fig. 45). Since cast iron is weak in tension the failure in each case is by tension.

ILLUSTRATIVE PROBLEM

Problem 201. If in Fig. 207 the axial tensile load, P, is equal to 45,000 lb., the twisting moment, Qq, is equal to 30,000 lb.-in. and the diameter, d, is equal to 3 in., find the maximum normal and shearing unit-stresses and the directions of the planes on which these stresses occur.

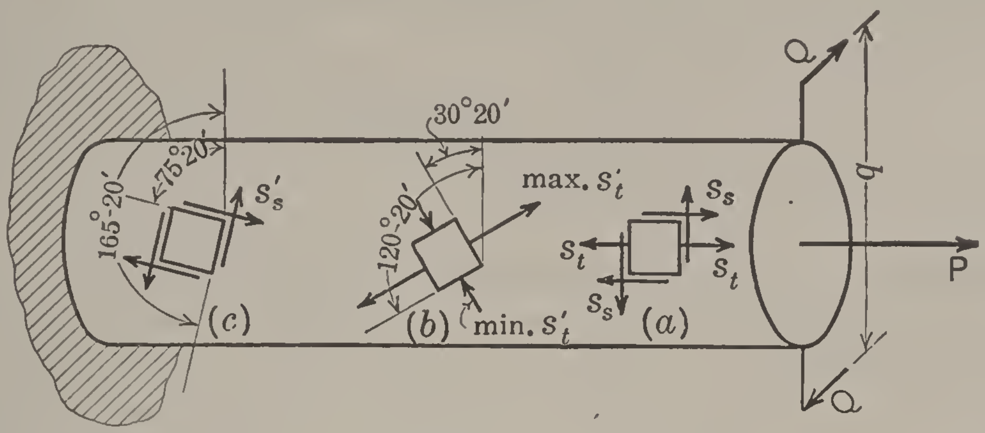

FIG. 207.—Stresses in bar subjected to combined tensile and torsional loads.

Solution.—The unit-stress, s_t, caused by the load P is

$$s_t=\frac{P}{a}=\frac{45,000}{\frac{\pi(3)^2}{4}}=6360 \text{ lb. per sq. in.}$$

and the shearing unit-stress, s_s, caused by the torsional moment T or Qq is

$$s_s=\frac{Tc}{J}=\frac{30,000\times 1.5}{\frac{\pi(3)^4}{32}}=5660 \text{ lb. per sq. in.}$$

The planes on which these stresses occur at an outer fiber of the shaft, are shown in Fig. 207(a).

The principal stresses, s'_t and s'_c, and the maximum shearing unit-stresses, s'_s, resulting from the above stresses are,

$$\text{max. } s'_t=\tfrac{1}{2}s_t+\tfrac{1}{2}\sqrt{s_t^2+4s_s^2}=\tfrac{1}{2}\cdot 6360+\tfrac{1}{2}\sqrt{(6360)^2+4(5660)^2}$$

$$=3180+6480=9660 \text{ lb. per sq. in., tensile stress.}$$

$$\text{min. } s'_t=\tfrac{1}{2}s_t-\tfrac{1}{2}\sqrt{s_t^2+4s_s^2}=-3300 \text{ lb. per sq. in., compressive stress,}$$

$$s'_s=\pm\tfrac{1}{2}\sqrt{s_t^2+4s_s^2}=6480 \text{ lb. per sq. in.,}$$

and the planes on which the principal stresses occur make the angles θ' with the plane on which the stress s_t occurs, the values of θ' being found as follows:

$$\tan 2\theta'=\frac{2s_s}{s_t}=\frac{2\times 5660}{6360}=1.78.$$

Hence

$$2\theta' = 60° \ 40' \text{ or } 240° \ 40',$$

and

$$\theta' = 30° \ 20' \text{ or } 120° \ 20'.$$

as indicated in Fig. 207(*b*). Further, the angles θ'_s that the planes of maximum shear make with the plane on which s_t occurs are

$$\theta'_s = \theta' + 45°$$

$$= 75° \ 20' \quad \text{and} \quad 165° \ 20'.$$

These planes are shown in Fig. 207(*c*).

Problem 202. A pressure P of 10,000 lb. on the crank pin of the steel crankshaft shown in Fig. 208(*a*) is required to turn the shaft at constant speed when the shaft is subjected to a constant resisting torque. If the diameter, d, of the shaft is 4 in., find the maximum combined normal and shearing unit-stresses at the section AB. Also find the ratios of the maximum normal and shearing stresses to the corresponding yield points of the material if the tensile yield point is 40,000 lb. per sq. in. P acts perpendicular to the pin and to the crank.

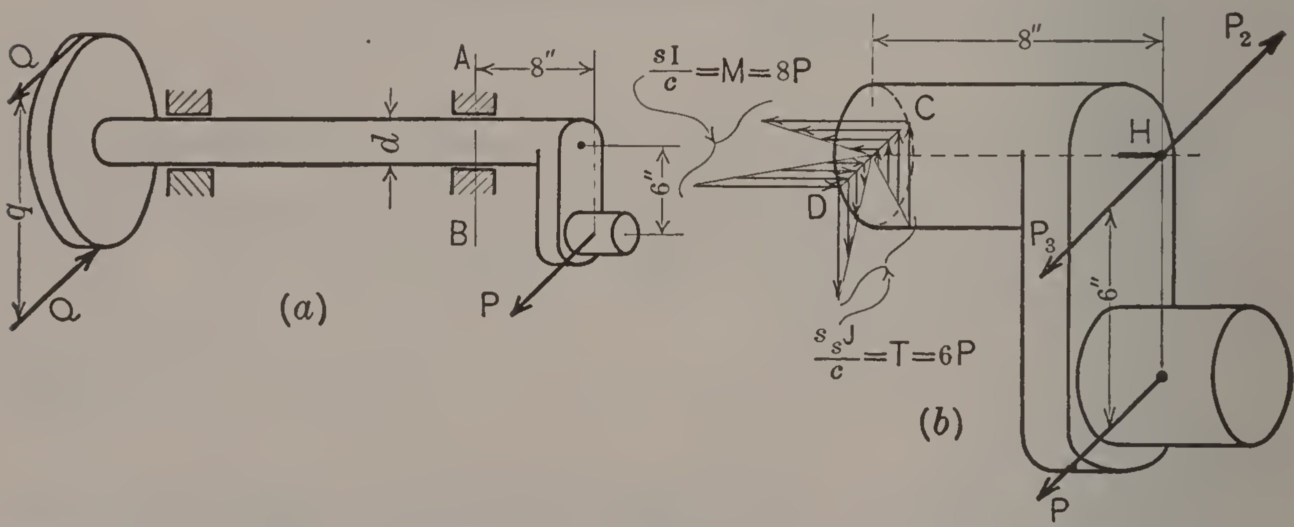

FIG. 208.—Stresses in crankshaft.

Solution.—By considering a free body diagram of the part of the shaft to the right of section AB, it is evident that the only external force acting on this part is P, and hence the stresses at section AB are caused by P and must be such as to hold P in equilibrium. Now by introducing two equal, opposite, and colinear forces P_2 and P_3 at H (Fig. 208*b*), P is resolved into a bending load P_3 and a torsional couple P, P_2; and these three forces cause the same stresses on the section AB as the original single force P. Thus the shaft to the right of the section AB is subjected to combined bending and torsional loads.

The bending moment at section AB due to the load P_3 is $8P$, and is held in equilibrium by the resisting moment $\frac{sI}{c}$ on the section AB as shown in Fig.

208(b). Thus the compressive unit-stress s at D (or the tensile unit-stress at C) is

$$s=\frac{Mc}{I}=\frac{8\times 10{,}000\times 2}{\frac{\pi(4)^4}{64}}=12{,}730 \text{ lb. per sq. in.}$$

The twisting moment due to the forces P and P_2 is $6P$ and is held in equilibrium by the shearing resisting moment $\frac{s_s J}{c}$ on section AB as shown in Fig. 208(b). Thus the shearing unit-stress s_s at D (or at C) is

$$s_s=\frac{Tc}{J}=\frac{6\times 10{,}000\times 2}{\frac{\pi(4)^4}{32}}=4770 \text{ lb. per sq. in.}$$

The maximum combined normal unit-stress at C and at D, then, is

$$s'=\tfrac{1}{2}\sqrt{s^2+4s_s{}^2}=\tfrac{1}{2}\times 12{,}730+\tfrac{1}{2}\sqrt{(12{,}730)^2+(4770)^2}$$

$$=6365+7935=14{,}300 \text{ lb. per sq. in.}$$

This is a compressive stress at D and a tensile stress at C. The maximum combined shearing unit-stress at C and at D is

$$s'_s=\tfrac{1}{2}\sqrt{s^2+4s_s{}^2}=7935 \text{ lb. per sq. in.}$$

If the tensile yield-point of the steel is 40,000 lb. per sq. in. and the shearing yield-point is six-tenths of the tensile yield-point (as found from tests, see Art. 140), then the maximum tensile stress is approximately 0.36 of the tensile yield-point, and the maximum shearing stress is 0.33 of the shearing yield-point.

PROBLEMS

203. If the values of P and Q in Fig. 205 (a) are 8000 lb. and 6000 lb. respectively, and the diameter of the bolt is $\frac{3}{4}$ in., what are the maximum normal (tensile) and shearing unit-stresses developed in the bolt? If the bolt is made of steel having a tensile yield-point of 45,000 lb. per sq. in. and a shearing yield-point equal to 0.6 of the tensile yield-point, what are the ratios of the maximum tensile and shearing stresses to the corresponding yield-points.

Ans. 0.56; 0.60.

204. A steam turbine drives an electric generator as indicated in Fig. 209. The diameter, d, of the shaft is 6 in. The weight of the generator is 15 tons.

The steam turbine delivers 1200 horse-power to the shaft and rotates at 800 r.p.m. Find the maximum normal and shearing unit-stresses developed in the shaft.

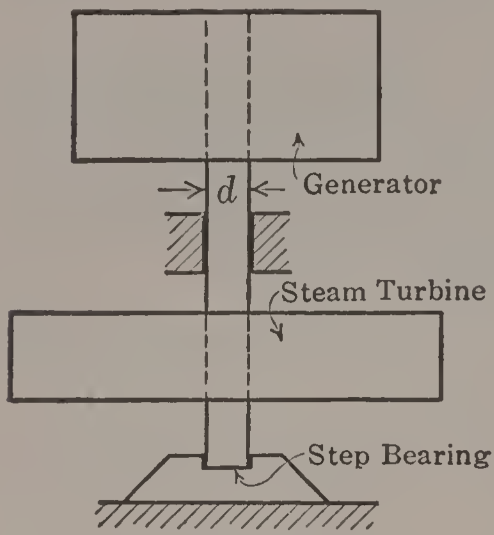

FIG. 209.

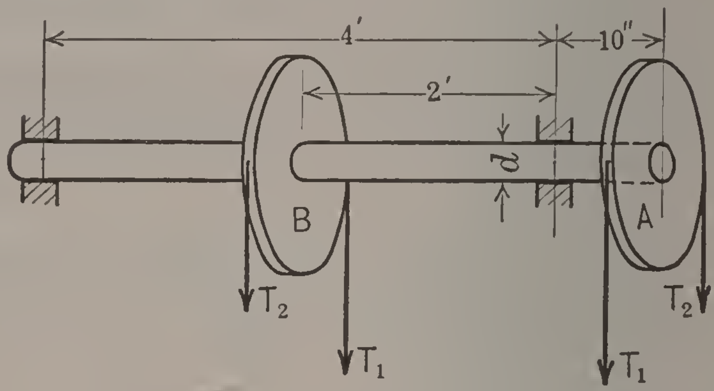

FIG. 210.

205. Two pulleys, A and B, are mounted on a shaft as shown in Fig. 210. The driving pulley, A, transmits 10 h.p. to the shaft and the driven pulley B. It turns the shaft at 100 r.p.m. The smaller belt tension, T_1, is 320 lb. and the diameter, d, of the shaft is 2 in. and the diameter of each pulley is 2 ft. Find the maximum normal and shearing unit-stresses in the shaft.

Ans. $s'_t = 15{,}800$ lb. per sq. in.; $s'_s = 8400$ lb. per sq. in.

206. The crank-pin pressure, P (Fig. 211*b*), perpendicular to the crank is 5000 lb. when the shaft is turning against a constant resisting moment Qq. The diameter of the shaft at A, B, and C, is 3 in. Find the reactions R_1 and R_2 of the bearings (which are assumed to be pivot bearings) and also the maximum normal and shearing unit-stresses at A, B, and C.

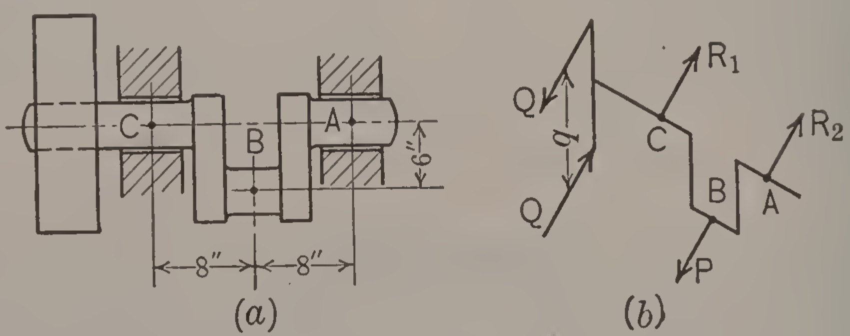

FIG. 211.

207. If a shaft is subjected to combined torsion and bending, and the shearing yield-point of the material is six-tenths of the tensile yield-point, prove that the ratio of the shearing unit-stress s'_s to the shearing yield-point will be equal to the ratio of the tensile unit-stress s'_t to the tensile yield-point when the ratio of the twisting moment T to the bending moment M is equal approximately to 0.9.

208. A shaft 4 in. in diameter is subjected to an axial end thrust of 12 tons, and also a bending moment and a twisting moment, the twisting moment being equal to one-half the bending moment. The maximum normal unit-stress is 10,000 lb. per sq. in. and the shaft rotates at 100 r.p.m. What horsepower does the shaft transmit?

209. A shaft 5 in. in diameter resists a bending moment of 120 ton-in. and a twisting moment of 70 ton-in. Find the magnitude of the maximum normal and shearing unit-stresses.

Ans. $s'_t = s'_c = 10.5$ ton per sq. in.; $s'_s = 5.66$ ton per sq. in.

210. In which of the two shafts described below is the greater normal unit-stress developed and in which is the greater shearing unit-stress developed? Calculate the maximum unit-stresses in each shaft.

(1) A 4-in. shaft subjected to a twisting moment of 40 ton-in. and a bending moment of 32 ton-in.

(2) A 2-in. shaft subjected to a twisting moment of 7.0 ton-in. and a bending moment of 20 ton-in.

110. Maximum Normal Stress Occurs on Plane of Zero Shearing Stress. Ellipse of Stress.—In Art. 109 it was stated that the maximum and minimum normal stresses occur on planes on which the shearing stress is zero. This fact will here be proved.

In Fig. 212 let ABC be an elementary prism at a point in any body; that is, AB, AC, and BC, are three planes passing through a given point in the body. Let the dimension of the prism perpendicular to the paper be unity. Further, let the planes AB and AC be the planes on which normal stresses only occur; the unit-stresses on these planes are denoted by s_1 and s_2 respectively. On the plane BC there will occur both a shearing unit-stress and a normal unit-stress; let the resultant or actual unit-stress on the face BC be denoted by s_r, and let the coordinates of the extremity of the vector s_r be x and y.

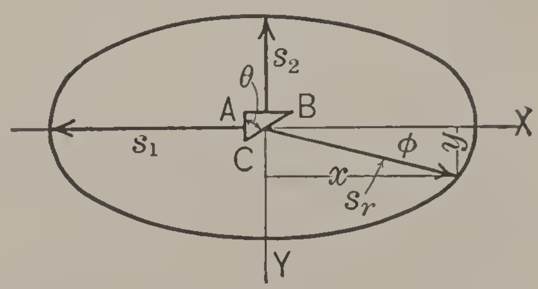

FIG. 212.—Ellipse of stress.

Now since the forces acting on the prism are in equilibrium, the sum of the x-components and of y-components must equal zero. Hence,

$$(\overline{BC}s_r)\cos\phi = ACs_1, \quad . \quad . \quad . \quad . \quad . \quad (203)$$

and

$$(\overline{BC}s_r)\sin\phi = ABs_2, \quad . \quad . \quad . \quad . \quad . \quad (204)$$

but $s_r \cos \phi = x$, $s_r \sin \phi = y$, $\frac{AC}{BC} = \cos \theta$, and $\frac{AB}{BC} = \sin \theta$. Hence (203) and (204) may be written as follows:

$$x = s_1 \cos \theta \quad \text{or} \quad \frac{x}{s_1} = \cos \theta,$$

and

$$y = s_2 \sin \theta \quad \text{or} \quad \frac{y}{s_2} = \sin \theta,$$

from which the following equation is obtained,

$$\frac{x^2}{s_1{}^2} + \frac{y^2}{s_2{}^2} = \cos^2 \theta + \sin^2 \theta = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (205)$$

This is an equation of an ellipse having s_1 and s_2, the principal unit-stresses, as the semi-axes, and hence the locus of the extremity of the vector representing the unit-stress on the face BC as this face is assumed to turn through 360° is an ellipse. This ellipse is called the *ellipse of stress*.

Now, as may be seen from inspection, the unit-stress s_r has its maximum or minimum value when it is one of the semi-axes of the ellipse, that is, when it occurs on a plane on which there is no shearing stress.

Planes of Maximum Shearing Stress.—In Art. 20 it was shown that the shearing unit-stress, s'_s, on the oblique plane BC (Fig. 212) is

$$s'_s = \tfrac{1}{2}(s_1 + s_2) \sin 2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (206)$$

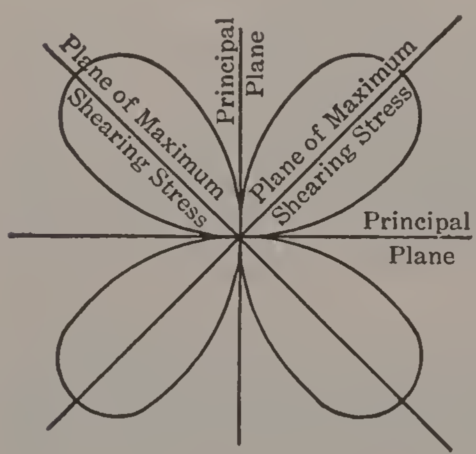

FIG. 213.—Planes of maximum shearing stress bisect angles between principal planes.

The graph of this equation is given in Fig. 213 in polar coordinates, s'_s being plotted as a radius vector and θ as the angle. This graph shows that the shearing unit-stress is a maximum on planes making angles of 45° with the planes on which the principal stresses s_1 and s_2 occur and that the shearing unit-stress is zero on the principal planes.

111. Diagonal Tensile Stress in a Beam.—The normal unit-stress on any right cross-section of a beam, such as section AA, Fig. 214, is the greatest at the outermost fiber and is found from the flexure formula,

$$s=\frac{Mc}{I},$$

where c is the distance from the neutral axis to the most remote fiber and M is the bending moment at the section considered. And since the shearing stress at the outermost fiber is zero (Art. 40), the stress s is the maximum normal stress at this point of the beam as is indicated at the outer fibers in section AA' of Fig. 214.

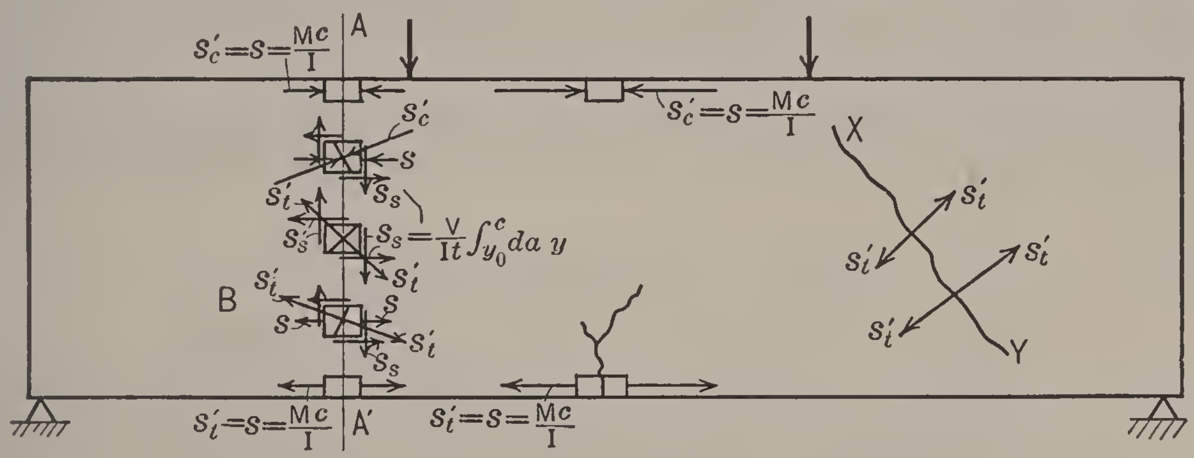

FIG. 214.—Resultant stresses in one section of beam.

At a point in the same section but nearer to the neutral axis, as at B, the normal stress on the vertical plane is also found from the same expression, in which c is the distance from the neutral axis to the point B, but this normal unit-stress is not the maximum normal unit-stress at this point since it is combined with shearing stresses on the vertical and horizontal planes, the values of which are

$$s_s=\frac{V}{It}\int_{y_0}^{c} yda \quad \text{(see Art. 40).}$$

Thus the resulting maximum normal unit-stress at B is

$$s'=\tfrac{1}{2}s+\tfrac{1}{2}\sqrt{s^2+4s_s{}^2} \quad \text{(see equation 202),}$$

in which s, and hence s', are tensile stresses at the point B. Further, the plane on which the stress s' occurs is inclined to the vertical cross-section on which s occurs as is indicated in Fig. 214, the stress s' is, therefore, frequently called the *diagonal tensile unit-stress* or briefly *diagonal tension.*

Now in most beams the diagonal tensile unit-stress at any point in a cross-section is less than the tensile stress at the outermost fiber, and hence need not be found. In reinforced concrete beams, however, the diagonal tensile stress is often of importance since concrete is very weak in tension and, if the beam has only longitudinal reinforcing bars in the lower part of the beam, the diagonal tension may cause cracking of the beam in that portion of the beam where the shearing stresses are relatively large; the diagonal tension crack for the beam loaded as shown in Fig. 214 would occur along a line similar to XY.

The relative directions of the maximum normal stresses at various points along a section AA' are shown in Fig. 214, and the relation of these to the diagonal tension crack XY is also shown. In the central portion of the beam where no shear exists the cracks are approximately vertical as indicated in Fig. 214.

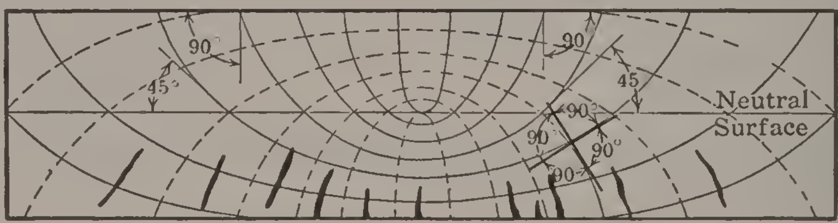

Fig. 215.—Directions of resultant stresses in beam.

The manner in which the directions of the principal stresses vary in a simple beam are shown in Fig. 215; the full lines refer to tensile stresses and the dotted lines to compressive stresses.

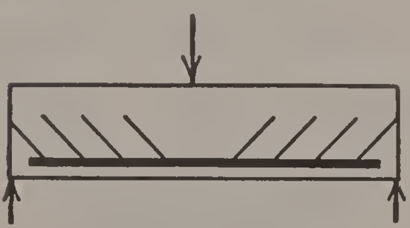

Fig. 216.—Steel reinforcing bars for resisting diagonal tensile stresses.

In order to prevent failure of concrete beams by diagonal tension, some of the longitudinal bars are usually bent up at the ends as indicated in Fig. 216 so that the bars have approximately the same direction as the diagonal tensile stresses.

A diagonal tension failure is frequently called a shear failure and the diagonal reinforcing bars are frequently called shear bars but it should be kept clearly in mind that the failure is due to tensile stresses and the bars are stressed in tension.

PROBLEMS

211. The load P acting on the short cantilever beam (Fig. 217) is 800 lb. (a) Find the unit-stress at point A normal to the vertical section. (b) Find the horizontal and vertical shearing unit-stresses at A. (c) Find the principal unit-stresses at A.

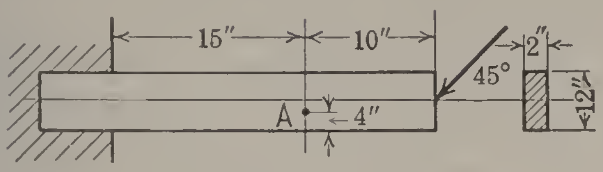

FIG. 217.

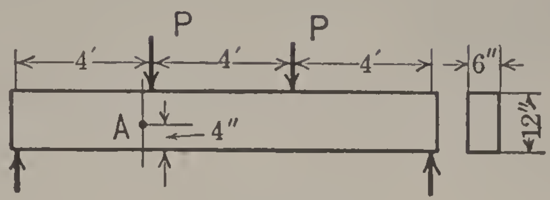

FIG. 218.

212. The loads P acting on the simple beam shown in Fig. 218 are each equal to 3800 lb. (a) Find the diagonal tensile unit-stress at the point A.

112. Expression for Eϵ.—As stated in Art. 22, when principal unit-stresses s_1 and s_2 occur on two mutually perpendicular planes at a point in a body the maximum unit-deformation, ϵ, at that point is in the direction of the larger unit-stress and is equal to

$$\epsilon=\frac{s_1}{E}-m\frac{s_2}{E}, \qquad (207)$$

in which s_1 is the larger stress and m is Poisson's ratio (Art. 21). When s_1 is a tensile stress ϵ is a unit-elongation, and when s_1 is a compressive stress ϵ is a unit-shortening. Further, when s_2 is opposite in sign to s_1 it is regarded as negative.

Now, as shown in Art. 109, when a body is subjected to loads that cause shearing stresses on two mutually perpendicular planes and a normal stress on one of the planes, there occur principal stresses on certain other planes. If, for example, loads cause a tensile stress combined with shear as in the bolt of Fig. 219, the maximum unit-deformation is an elongation in the direction of the maximum principal stress s'_t and its value is

$$\epsilon=\frac{s'_t}{E}-m\frac{s'_c}{E}, \qquad (208)$$

or

$$E\epsilon=s'_t+ms'_c,$$

since s'_c is negative.

By substituting in the above equations the values of s'_t and s'_c found in Art. 109 the following equations are obtained:

$$E\epsilon=\tfrac{1}{2}s_t+\tfrac{1}{2}\sqrt{s_t^2+4s_s^2}+m(-\tfrac{1}{2}s_t+\tfrac{1}{2}\sqrt{s_t^2+4s_s^2})$$

$$=\tfrac{1}{2}(1-m)s_t+\tfrac{1}{2}(1+m)\sqrt{s_t^2+4s_s^2}. \qquad (209)$$

For steel the value of m found from experimental results (see Art. 21) is 0.25 to 0.30. Thus, for steel, the above equation becomes,

$$E\epsilon=\tfrac{3}{8}s_t+\tfrac{5}{8}\sqrt{s^2{}_t+4s_s{}^2},\quad \text{when}\quad m=0.25,\ \ldots \quad (210)$$

and

$$E\epsilon=0.35s_t+0.65\sqrt{s_t{}^2+4s_s{}^2}\quad \text{when}\quad m=0.30.\ . \quad (211)$$

Now $E\epsilon$ may be called the *equivalent uni-directional unit-stress*,[3] since, if ϵ were caused by a load that developed a stress in one direction only (uni-directional stress), the value of $E\epsilon$ would then be numerically equal to the actual unit-stress; thus, when a bar is pulled by an axial load P in a testing machine, the value of the unit-stress, s, is fixed by the conditions of equilibrium and is equal to $\frac{P}{a}$, but experiment shows that s is also numerically equal to $E\epsilon\left(\frac{s}{\epsilon}=E\right)$. Likewise, when two normal stresses at right angles

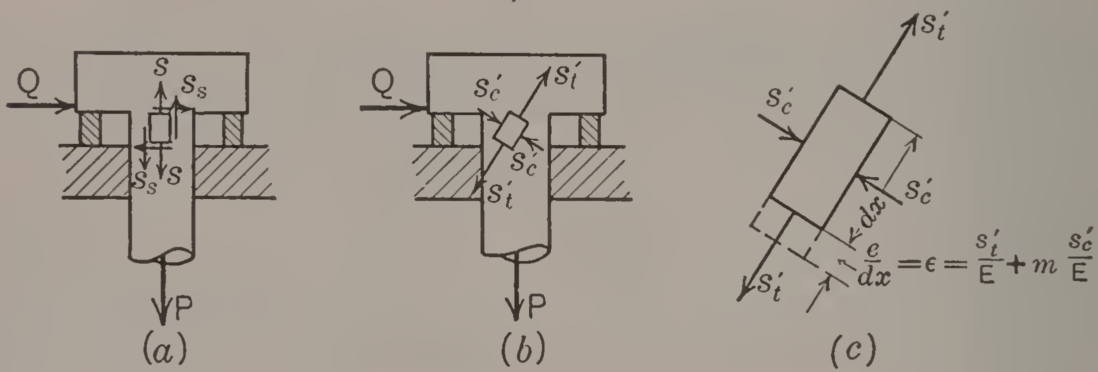

FIG. 219.—Strain due to principal stresses.

to each other are developed as in Fig. 219(b) and (c) each stress is fixed solely by the conditions of equilibrium but the value of $E\epsilon$ is not equal to either of the unit-stresses (see also Art. 22). The significance and use of the quantity $E\epsilon$ will be discussed in the following article.

113. Theories of Failure of Elastic Action and Their Application.—If a bar is subjected to a gradually increasing axial tensile load *in one direction only*, the material, when the load reaches a certain value, will begin to acquire permanent deformation and

[3] Other names, such as simple equivalent stress, strain-stress, etc., have been used. However, it is important to note that $E\epsilon$ is not a real unit-stress (force per unit area) in the material but merely a quantity expressed in the same units (lb. per sq. in) as a unit-stress.

hence to fail in elastic action, and the reason for the failure may be (1) because the normal (tensile) unit-stress $\left(s_t = \frac{P}{a}\right)$ reaches the tensile elastic limit, s_e (or proportional limit, see Art. 137c), or (2) because the shearing unit-stress reaches the shearing elastic limit; this requires that the shearing elastic limit shall be equal to or less than one-half the tensile elastic limit since, as shown in Art. 16, the maximum shearing unit-stress, on a 45-degree oblique plane, is one-half the tensile unit-stress, or (3) because the unit-deformation reaches a value which cannot be exceeded without having part of the deformation plastic and hence permanent, or (4) because the energy absorbed by (or the work done on) the material per unit volume reaches a value $\left(\frac{1}{2}\frac{s_e^2}{E} \text{ or } \frac{1}{2}\frac{s_s^2}{E_s}, \text{ see Art. 23}\right)$ that can not be exceeded unless the material is given a plastic deformation. It is impossible to determine from a simple tension test, that is, from a test in which the stress is in one direction only, what is the real cause of the beginning of inelastic action in a material, because the above limiting values occur simultaneously. The four main theories suggested above may be stated briefly as follows:

1. The *maximum normal stress* theory, often called *Rankine's* theory, states that elastic action at any point in a material ceases, or inelastic action begins, *only* when the maximum normal (tensile) stress on a certain plane passing through the point reaches a value equal to the tensile elastic limit found in a simple tension test, regardless of the normal or shearing stresses that occur on other planes through the point. Thus, if the block in Fig. 220(*a*) reaches its elastic limit when subjected to the stress s_1, the elastic limit will still be s_1 even if the block is subjected to the stress s_2 (Fig. 220*b*) in addition to s_1.

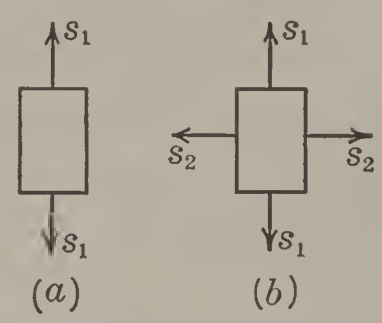

FIG. 220.

2. The *maximum strain* theory, often called *Saint Venant's* theory, states that inelastic action at any point in a body due to any combination of stresses at the point begins *only* when the maximum unit-elongation ϵ at the point reaches a value equal to that which occurs when inelastic action begins in a bar subjected to an axial tension test; which is the value of ϵ that occurs simul-

taneously with the tensile elastic limit s_e of the material. Now since $s_e = E\epsilon$ when the stress is applied in one direction only, the maximum strain theory states that inelastic action begins at any point in a body when the value of $E\epsilon$ at that point reaches a value equal to the tensile elastic limit of the material.

For example, according to this theory inelastic action in the block of Fig. 220(*a*) begins when s_1 equals the elastic limit of the material, but inelastic action in the block of Fig. 220(*b*) does not begin until $E\epsilon$ in the direction of s_1 is equal to the elastic limit of the material, and this value of $E\epsilon$ will not occur until s_1 in Fig. 220(*b*) is larger than s_1 of Fig. 220(*a*). For, the unit-deformation in the direction s_1 of the block in Fig. 220(*a*) is made less by the amount $m\frac{s_2}{E}$ when the stress s_2 is applied as in Fig. 220(*b*) (see Art. 22), and hence the stress s_1 must be increased before the limiting unit-deformation is reached.

3. The *maximum shearing stress* theory, sometimes called Guest's theory, states that inelastic action at any point in a body begins *only* when the maximum shearing unit-stress on some plane through the point reaches a value equal to the shearing elastic limit found by testing the material in simple shear as in a torsion test (see Fig. 258 and Art. 140). And, as noted above, this theory when applied to a simple tension test of a bar requires that the shearing elastic limit shall be one-half the tensile elastic limit.

4. The *maximum energy* or *maximum resilience* theory, proposed by Haigh, states that inelastic action at any point in a body due to any combination of stresses begins only when the energy per unit volume absorbed by the material at that point equals the energy per unit volume absorbed by a bar when stressed to the elastic limit in a simple tensile test. As shown in Art. 23, the value of this maximum energy per unit volume is $\frac{1}{2}\frac{s_e^2}{E}$.

Significance of the Theories.—It must be kept clearly in min that the actual unit-stresses at a point in a body are fixed solely by the external forces, the stresses being such as to hold the external forces in equilibrium, and it is assumed that the stresses are known or can be found. The only uncertainty which the theories attempt to explain is the state or condition developed in the material that causes inelastic action in the material to start.

Several experimental investigations have been carried out on ductile material, such as steel and brass, with combinations of stresses designed to make clear the cause of inelastic action, and although the results are not entirely concordant, the best conclusion appears to be that in *ductile* metals inelastic action begins when the shearing unit-stress reaches the shearing elastic limit of the material unless the limiting value of $E\epsilon$ is reached before the shearing elastic limit is reached, in which case inelastic action begins when the limiting value of $E\epsilon$ is developed. The tests indicate, however, that the shearing elastic limit is somewhat greater than one-half of the tensile elastic limit as demanded by the maximum shear theory, the value of the shearing elastic limit for steel being close to six-tenths of the tensile elastic limit. The maximum energy theory also agrees fairly well with tests results for the combinations of stress used in the tests.

In many of the problems discussed in the preceding chapters all these theories lead to the same results. In other problems the difference in the results obtained by the four theories are not important, and in some problems the assumptions made as to the distribution of loads and stresses (see Appendix III) contain errors that would not justify strong reliance on the results from one theory in preference to those from another. Partly for these reasons and partly due to long usage and to the too-frequent practice of assuming that the low value of the working stress is sufficient to compensate for all uncertainties, the maximum stress theory has been most commonly used, although the maximum shear theory, for ductile material, has gained rather wide acceptance in recent years. The maximum strain theory has been used extensively in the design of guns.

The method of applying the theories to a problem involving combined bending and torsion is given below.

Application of Theories of Failure of Elastic Action.—As stated in the preceding article, the dimensions that should be assigned to a member which is subjected to loads causing combined stresses depend on the theory held concerning the cause of the breakdown of elastic action. This fact will be illustrated in the solution of a problem of the design of a member subjected to loads that cause the combination of stresses discussed in Art. 109; namely, shearing stresses on two planes at right angles to each other and a normal stress on one of the planes. In the following problem this combination of stresses is caused by combined bending and torsion.

ILLUSTRATIVE PROBLEM

Problem 213. The crank-shaft shown in Fig. 208 is made of steel having a tensile and compressive proportional limit of 40,000 lb. per sq. in. Determine, by each of the four theories of failure, the diameter, d, the shaft should have to resist the load P without developing more than one-half of the maximum elastic strength of the material. Tests show that the shearing proportional limit of steel is approximately 0.6 of the tensile proportional limit.

Solution.—As noted in Problem 202, the twisting moment T and the bending moment M have the following values:

$$T=6P=60{,}000 \text{ lb.-in.} \quad \text{and} \quad M=8P=80{,}000 \text{ lb.-in.};$$

and the shearing and normal unit-stresses due, respectively, to T and M are

$$s_s=\frac{Tc}{J}=\frac{16T}{\pi d^3} \quad \text{and} \quad s=\frac{Mc}{I}=\frac{32M}{\pi d^3}.$$

(*a*) *Maximum Normal Stress Theory.*—The maximum normal unit-stress resulting from the unit-stresses s_s and s, as shown in Art. 109, is

$$s'=\tfrac{1}{2}s+\tfrac{1}{2}\sqrt{s^2+4s_s^2},$$

and according to the maximum stress theory inelastic action in the material begins when s' exceeds 40,000 lb. per sq. in Hence the working unit-stress is 20,000 lb per sq. in. Therefore,

$$20{,}000=\frac{1}{2}\frac{32M}{\pi d^3}+\frac{1}{2}\sqrt{\left(\frac{32M}{\pi d^3}\right)^2+\left(\frac{2\times16T}{\pi d^3}\right)^2}$$

$$=\frac{16}{\pi d^3}(M+\sqrt{M^2+T^2}),$$

whence

$$d^3=\frac{16}{\pi 20{,}000}(80{,}000+\sqrt{(80{,}000)^2+(60{,}000)^2}),$$

and

$$d^3=\frac{8}{\pi}(8+\sqrt{(8)^2+(6)^2})=45.8 \text{ in.}^3,$$

Therefore

$$d=3.57 \text{ in.}$$

(*b*) *Maximum Strain Theory.*—According to the maximum strain theory, inelastic action in the material begins when the maximum value of $E\epsilon$ becomes 40,000 lb. per sq. in. And, from Art. 112 the maximum value of $E\epsilon$, assuming m equal to 0.30, is given by the equation

$$E\epsilon=0.35s+0.65\sqrt{s^2+4s_s^2}.$$

The working value of $E\epsilon$ is $\frac{1}{2}\times 40{,}000$. Thus,

$$20{,}000=\frac{32}{\pi d^3}\,(0.35M+0.65\sqrt{M^2+T^2}).$$

Hence

$$d^3=\frac{16}{\pi}(0.35\times 8+0.65\times 10)=47.4\ \text{in.}^2$$

Therefore

$$d=3.62\ \text{in.}$$

(c) *Maximum Shearing Stress Theory.*—According to this theory inelastic action in the shaft begins when the maximum shearing unit-stress s'_s, as given by the equation

$$s'_s=\tfrac{1}{2}\sqrt{s^2+4s_s^2},$$

reaches the shearing proportional limit, that is, $0.6\times 40{,}000$ or 24,000 lb. per sq. in. The working unit-stress then is 12,000 lb. per sq. in. Hence,

$$12{,}000=\tfrac{1}{2}\sqrt{s^2+4s_s^2}$$

$$=\frac{16}{\pi d^3}\sqrt{M^2+T^2}.$$

Hence,

$$d^3=\frac{160}{12\pi}\sqrt{8^2+6^2}=\frac{1600}{12\pi}=42.5\ \text{in.}^3$$

Therefore,

$$d=3.48\ \text{in.}$$

(d) *Maximum Energy Theory.*—As shown in Art. 23 the expression for the energy absorbed by the material per cubic inch when subjected to the stresses s and s_s, as in the shaft here considered, is

$$U=\frac{1}{2}\,\frac{s^2}{E}+\frac{1}{2}\,\frac{s_s^2}{E_s}.$$

But, for steel $E_s=\frac{2}{5}E$ (Art. 5) and hence the above expression may be written

$$U=\frac{1}{2}\,\frac{s^2}{E}+\frac{5}{4}\,\frac{s_s^2}{E}.$$

Now, according to the maximum energy theory the maximum value of the energy U that can be absorbed per unit volume without causing inelastic action in the material is

$$U_{\text{max.}}=\frac{1}{2}\,\frac{s_e^2}{E}=\frac{1}{2}\,\frac{(40{,}000)^2}{30{,}000{,}000}=26.6\ \text{in.-lb. per cu. in.}$$

But since it is specified that the material shall be required to absorb $\frac{1}{2} \times 26.6 = 13.3$ in.-lb. per cu. in., when the shaft is subjected to the given load, we have,

$$13.3 = \frac{1}{2}\frac{s^2}{E} + \frac{5}{4}\frac{s_s^2}{E},$$

$$4 \times 13.3 \times E = 2s^2 + 5s_s^2$$

$$= 2\left(\frac{32M}{\pi d^3}\right)^2 + 5\left(\frac{16T}{\pi d^3}\right)^2,$$

$$(\pi d^3)^2 = 11{,}100,$$

$$d = 3.22 \text{ in.}$$

Thus, in this problem the maximum strain theory requires the largest shaft.

PROBLEMS

214. In the above problem change the length of the crank from 6 in. to 12 in. and find, by each of the four theories, the diameter of the shaft required.

215. Find, by all four theories of failure of elastic action, the area of cross-section that the steel bolt shown in Fig. 219(a) should have in order to resist a load P of 4000 lb. and a load Q of 3000 lb. without developing more than $\frac{1}{2}$ of the maximum elastic strength of the material. The bolt is made of steel having a tensile proportional limit of 30,000 lb. per sq. in. Use a value of 0.25 for Poisson's ratio.

216. Find, by all four theories of failure of elastic action, the diameter of the shaft shown in Fig. 202, if the load P is 8000 lb. and the torsional moment, Qq, is 16,000 lb.-in. Assume that the shaft, in resisting the loads, develops 0.65 of the maximum elastic strength of the material. The shaft is made of steel having a tensile elastic limit of 60,000 lb. per sq. in. Use a value of 0.25 for Poisson's ratio.

217. Prove that, when a cylindrical shaft is subjected to combined bending and torsion, the ratio of the maximum shearing stress s'_s to the shearing yield point will be equal to the ratio of $E\epsilon$ to the tensile yield-point when the ratio of the twisting moment, T, to the bending moment, M, is approximately equal to 1.62, provided that the shearing elastic limit is 0.6 of the tensile elastic limit. (See Problem 207 for the ratio of T to M according to the maximum stress theory.)

218. A circular steel shaft having a tensile elastic limit of 36,000 lb. per sq. in. is subjected to a bending moment of 50 ton-in. and a twisting moment of 50 ton-in. Assume that the shearing elastic limit is 0.6 of the tensile elastic limit. (*a*) Compute by each of the four theories of failure of elastic action the diameter the shaft must have if it develops $\frac{1}{3}$ the maximum elastic strength of the material. (*b*) Change the bending moment to 5 ton-in. and repeat the problem. (*c*) Change the twisting moment to 5 ton-in. leaving the bending moment at 50 ton-in. and again repeat.

Ans. (*a*) 4.68; 4.74; 4.65; 3.97. (*b*) 3.60; 3.84; 4.14; 3.39. (*c*) 4.40; 4.40; 4.14; 3.67.

CHAPTER XIII

IMPACT AND ENERGY LOADS

114. Introduction.—In the preceding chapters, the loads acting on the members were assumed to be gradually applied (static loads). To state the same idea in other words, the bodies that applied the loads were not in motion when they came in contact with the resisting member, and hence they delivered no kinetic energy to the resisting member.

Members of engineering structures and machines, however, frequently must resist loads that are applied by moving bodies, and the kinetic energy of these moving bodies must be absorbed by the resisting member, thereby developing stresses and deformations in the member.

Now there are two methods of determining the stresses and deformations developed in the resisting member: (1) To estimate the maximum pressure or force exerted by the moving body on the resisting member; this force is then considered to be a static load and is used in the equations developed in the preceding chapters. (2) To estimate the energy that is absorbed by the resisting member and from this value of the energy determine the stresses and deformations by use of the equations developed in this chapter.

In the first method the load is a force and is, of course, expressed in pounds, tons, etc.; it is called an impact load and the method is frequently called the *equivalent-static-load* method. Whereas, in the second method the load is considered to be a quantity of energy and is therefore called an *energy load*; it is expressed in foot-pounds, inch-tons, etc.

The determination of the stresses in structural and machine members that are subjected to impact or energy loads is less definite and satisfactory than in members subjected to static loads, for the reason, that, in general, the uncertainty as to the value of an impact load or of an energy load is greater than that of a static load. The design of members subjected to energy loads, therefore, is based more directly on the study of existing machines and structures than is the design of members subjected to static loads. However, in this connection it is well to recognize, as stated in Art. 7, that the static loads to which members in structures and machines are assumed to be subjected frequently are only rough approximations of the actual loads that the members are required to resist.

When the energy delivered to the resisting member is relatively small the equivalent-static-load method is the one more commonly used in determining the stresses in the member. Thus the load caused by a moving train on a bridge, by wind on a building, by a moving crowd of people on a stadium, etc., is assumed to be equivalent to a static load composed of the actual static or " dead " load plus a static load assumed to be equivalent (in producing stress) to the dynamic or impact effect of the load. This additional (equivalent) static load is frequently called the " live " load.

If, on the other hand, the energy delivered to the resisting member is relatively large the energy-load method may be the more useful one, for, it will be found that the dimensions of the resisting member and the properties of the material in the member that give it maximum resistance to an energy load are quite different from those that give the member maximum resistance to a static load.

115. Calculation of Energy Delivered to Resisting Member.— As a rule, only a part of the energy of the moving body that delivers the energy to the resisting member is absorbed or stored in the resisting member; some of the energy is spent in each of the following ways:

(*a*) In causing stresses and deformations throughout the moving body itself.

(*b*) In causing local stresses and deformation of both bodies at the surface of contact, especially if the velocity of the moving body is large when it comes in contact with the resisting body.

(*c*) In overcoming the inertia resistance of the resisting member, and

(*d*) In deforming and in moving the external supports to which the resisting member is attached.

Thus, for example, an airplane when landing possesses kinetic energy which is delivered mainly to the wheels and axle on which the wheels are mounted, and these parts are constructed so that they can absorb a large amount of the energy. But, a considerable part of the energy of the airplane is absorbed as stated under (*a*) and (*d*) above, that is, in causing deformation of the frame of the airplane and in deforming the tires and spokes of the wheels and in causing depressions in the ground.

Kinetic Energy of Bodies.—If the moving body that delivers the energy has a motion of translation or of rotation the kinetic energy of the body may be found from the expressions given below; from this kinetic energy must be subtracted the amount of energy assumed to be lost, in order to obtain the energy load absorbed by the resisting member.

Translation.—The kinetic energy E_k, of a body having a motion of translation is

$$E_k=\frac{1}{2}Mv^2=\frac{1}{2}\frac{W}{g}v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (212)$$

in which v is the velocity of the body and M is the number of units of mass in the body (found by dividing the weight, W, of the body by g, the acceleration which the weight causes when it is the only force acting on the body $\left(M=\frac{W}{g}\right)$. When W is expressed in pounds, g in feet per second per second, and v in feet per second, E_k will be expressed in foot-pounds. Further, if the body acquires the velocity v by falling freely from rest through a distance of h feet, then v^2 is equal to $2gh$ and hence $E_k=Wh$. (The value of g is 32.2 ft. per sec.^{2}, approximately).

Rotating.—If the body that delivers the energy load has a motion of rotation about a fixed axis its kinetic energy E_k may be found from the expression

$$E_k=\tfrac{1}{2}I_0\omega^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (213)$$

in which ω is the angular velocity of the body and I_o is the moment of inertia of the body about the axis of rotation; I_o is expressed in terms of the mass $\left(\frac{W}{g}\right)$ of the body and the dimensions of the body. Thus, for a solid cylinder that rotates about its geometric axis $I_0=\frac{1}{2}\frac{W}{g}r^2$, where r is the radius of the cylinder. If W is expressed in pounds, g in feet per second per second, the dimensions of the body in feet, and ω in radians per second, E_k will be expressed in foot-pounds. The values of I_0 for bodies of other shapes may be found in books on analytical mechanics and in engineers' handbooks.

116. Stress in Bar Due to Axial Energy Load.—Let an energy load U be applied to a bar so that the bar is subjected to an axial tensile stress. Such a load occurs in machines of various types as will be discussed later; the method of estimating the value of U is given in the preceding article; namely, the value of U is the kinetic energy of the moving body less the part of this energy that is lost in ways stated in Art. 115.

The problem is to find the maximum unit-stress in the bar caused by the energy load U. The assumption [1] is made that a material when resisting an energy load acts in the same way as when resisting a gradually applied (static) load; namely, unit-stress is proportional to unit-deformation until the proportional limit is reached. Hence, the energy U stored in the bar when the unit-stress in the bar is s (which is equal to the work, w, done on the bar in causing the stress s, provided that s is not greater than the proportional limit) may be expressed as follows:

$$U = w = \tfrac{1}{2}Pe,$$

in which P is the final value of the gradually applied load and e is the total elongation of the bar. But since s is within the proportional limit we have $\frac{s}{\epsilon} = E$. Further, $s = \frac{P}{a}$ where a is the cross-sectional area of the bar. Hence,

$$U = \frac{1}{2}\frac{s^2}{E}al.$$

Now, since the energy stored in the bar is assumed to be independent of the way the energy is delivered to the bar, the relation between any energy load U and the unit-stress s is

$$U = \frac{1}{2}\frac{s^2}{E}al \quad \text{or} \quad s = \sqrt{\frac{2UE}{al}}, \quad . \quad . \quad . \quad . \quad (214)$$

[1] This assumption probably is not true if the load is applied with extreme suddenness and acts on the body only a very short period of time, causing a "momentary" stress. But with energy loads as they usually occur in engineering practice the assumption is probably justified although experimental verification is difficult to obtain.

in which s must not exceed the proportional limit of the material. If U is expressed in inch-pounds, E in pounds per square inch, a in square inches, and l in inches, s will be expressed in pounds per square inch. Equation (214) shows that the energy absorbed per unit-volume of the material, when stressed to the proportional limit, s_e, is $\frac{1}{2}\frac{s_e^2}{E}$; this value is represented by the area OCD in Fig. 221 and is called the *modulus of resilience* of the material (see Art. 23 and 146).

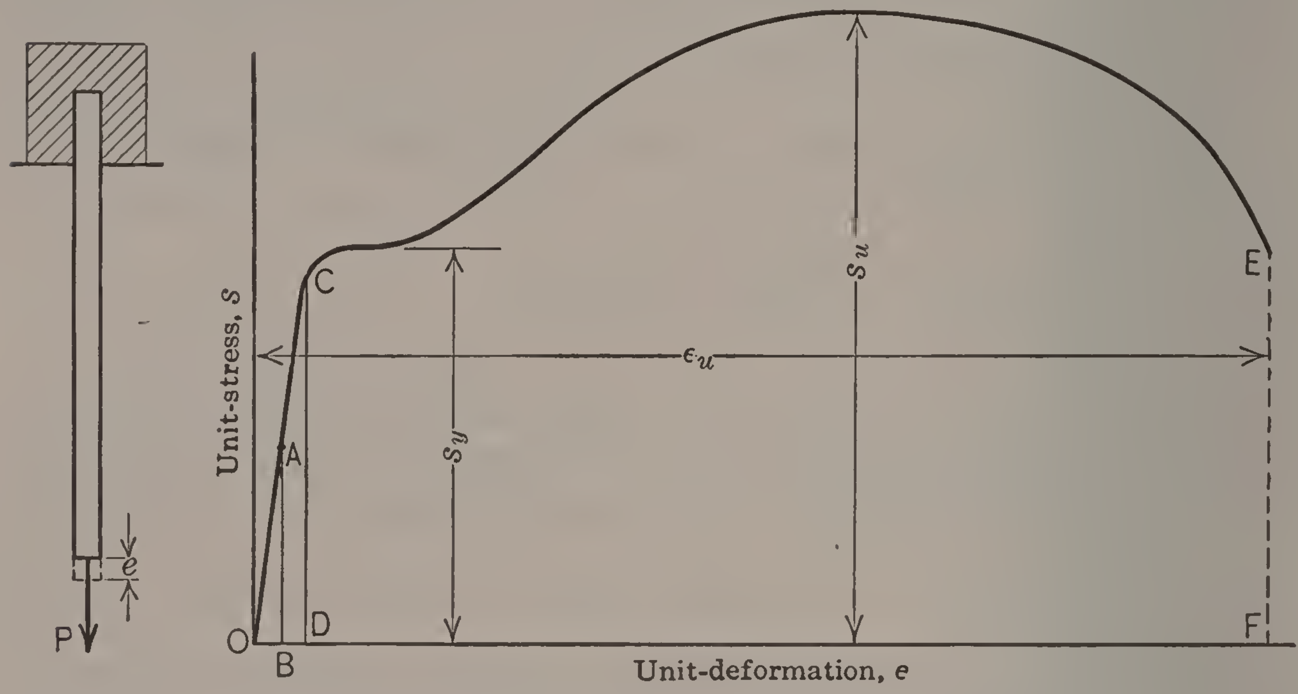

FIG. 221.—Stress-strain diagram for ductile steel.

The ideal material, then, for resisting energy loads in service in which the material must not incur permanent distortion, is one having a high modulus of resilience, that is, one having a high proportional limit and a low modulus of elasticity (see Art. 146 for further discussion).

Ultimate Energy Resistance of a Material. Toughness.—The ultimate energy resistance of a material is the maximum amount of work that can be done on the material per unit of volume, without causing the material to rupture. It is represented by the total area under the stress-strain curve,[1] area $OCEF$ in Fig. 221. Most of this work is dissipated in heat and in causing permanent deformation of the material, and only a very small part is stored

[1] It is probable that the ultimate energy resistance is greater than that represented by the area under the static stress-strain curve, and hence the error introduced is on the side of safety.

in the material as stress-energy that can be recovered when the stress is released. The property of a body by virtue of which work can be done on the body when stressed beyond its elastic limit is called *toughness*. A tough material is needed, therefore, to resist energy loads when the material in service is likely to be stressed beyond its yield-point.

For materials having a stress-strain diagram similar to that shown in Fig. 221 (that is, for ductile materials) the area under the curve is given approximately by the expression $\frac{s_y+s_u}{2}\epsilon_u$, and hence the ultimate energy load U the bar can resist is, approximately,

$$U=\frac{s_y+s_u}{2}\epsilon_u\cdot al, \quad . \quad . \quad . \quad . \quad . \quad (215)$$

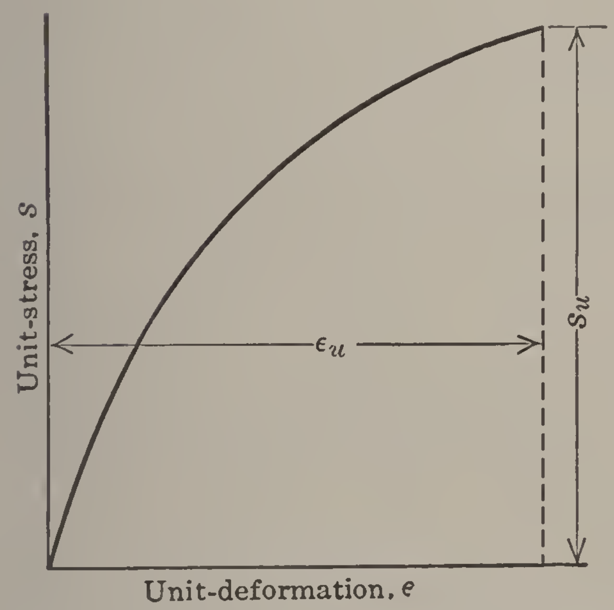

FIG. 222.—Stress-strain diagram for brittle material.

in which s_y and s_u are the yield-point and ultimate strength of the material, respectively, and ϵ_u is the ultimate unit-deformation. And, for material having a parabolic stress-strain diagram (Fig. 222) such as cast iron and concrete, the area under the curve, is approximately $\frac{2}{3}s_u\epsilon_u$, and hence the ultimate energy load a bar of such a material can resist is, approximately,

$$U=\tfrac{2}{3}s_u\epsilon_u. \quad . \quad . \quad . \quad . \quad . \quad . \quad (216)$$

117. Comparison of Effects of Static and Energy Loads.— The resistance offered by a bar to a *static* axial load, P, depends only on the maximum unit-stress developed, which occurs on the smallest cross-section ($P=as$), whereas the resistance of the bar to an energy load, U, as indicated by equation (214), depends not only on the maximum unit-stress, s, but also (1) on the distribution of the stress throughout the body, since the energy absorbed by a given unit of volume is $\frac{1}{2}\frac{s^2}{E}$, and hence depends on the degree to which that volume is stressed, and (2) on the number of units of volume (al) of material in the bar. The influence of these two factors may be shown as follows:

(1) The static strengths of the two cylindrical bars shown in Fig. 223 when subjected to axial loads are equal, since the smallest cross-sections are equal, and hence the loads P required to produce a given stress s in the bars are equal ($P=as$). The energy loads required to produce a given stress in the two bars, however, are very different; the bar having the constant diameter, and hence a uniform distribution of stress throughout the length of the bar, is able to absorb the greater amount of energy. For example, if the diameter of the upper half of the bar shown in Fig. 223(*b*) is reduced from $2d$ to d the area a (and also volume al) will, thereby, be decreased to one-fourth of the original area (and volume), and hence the unit-stress will be increased to four times the original value. Therefore, the energy absorbed per unit volume $\left(\frac{1}{2}\frac{s^2}{E}\right)$ by this upper half will be sixteen times as great as that absorbed per unit-volume when the diameter is $2d$. Thus, in the expression $\frac{1}{2}\frac{s^2}{E}al$ for the upper half of the bar, the factor $\frac{1}{2}\frac{s^2}{E}$ has been increased more than the factor al has been decreased; the total energy absorbed by the upper half of the bar being increased to four times its original value by reducing the diameter of the upper half from $2d$ to d.

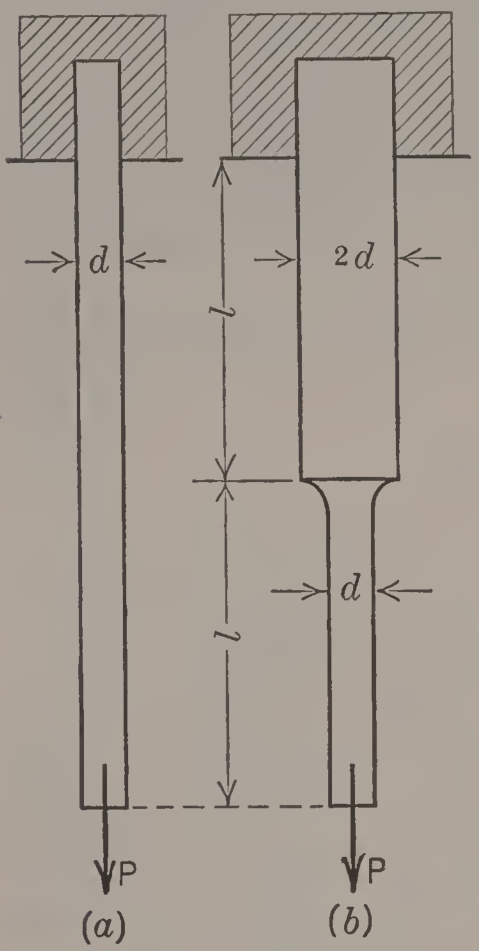

FIG. 223.—The bars have equal resistance to static loads, but not to energy loads.

This method of increasing the energy load that can be applied to a bar was brought forcibly to the attention of engineers in the early development of the Straight Line engine by Professor John E. Sweet. The bolts in the connecting-rod head when made in the usual form with full-sized shanks and threaded ends as in Fig. 224(*a*) frequently broke [3] in service due to the energy load

[3] As will be shown in the next chapter, failure of material is frequently due to the repeated application of a stress if the stress is above a limiting minimum value, and this was probably the cause of the failure of the bolts

delivered by the reciprocating parts of the engine. By turning down the bolts so that the area of cross-section of the shank was equal to that at the roots of the threads, as shown in Fig. 224(*b*), the difficulty was removed since most of the energy of the reciprocating parts was then absorbed in the shank of the bolt, and the amount

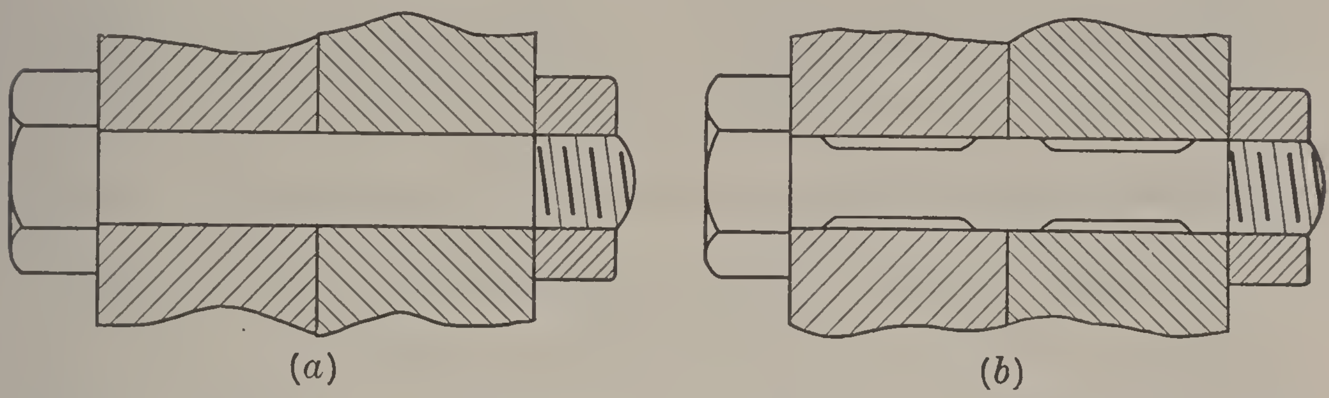

FIG. 224.—Energy resistance of bolt increased by decreasing diameter of shank.

absorbed at the roots of the thread was thereby greatly reduced with a corresponding reduction in the stress developed at the roots of the threads.

It will be seen, therefore, that uneven distribution of stress throughout a member may greatly reduce the energy load that can be applied to the body.

2. If the stress in a bar is distributed uniformly throughout the bar, that is, if the area of cross-section is constant, the energy load that the bar will resist may be increased by increasing the length of the bar since this increases the volume, each unit of volume absorbing the same amount $\left(\frac{1}{2}\frac{s^2}{E}\right)$ of energy.

This method of increasing the energy resistance of a member was clearly illustrated in the early development of rock drills.[4] The cylinder heads of the rock drills were attached to the cylinder as shown in Fig. 225(*a*) and since in the operation of the drill it is impossible to avoid occasional sharp blows of the piston on the cylinder head the bolts were subjected to severe energy loads and, as a consequence, broke. The trouble was remedied by using

here referred to. But, by reducing the stress in the bolts, as stated, the repetitions of this lower stress did not cause failure.

[4] Halsey, F. A. "Materials and Constructions for Resisting Shock," American Machinists, Sept. 9, 1915, p. 459.

bolts long enough to extend through both cover plates as shown in Fig. 225(*b*).

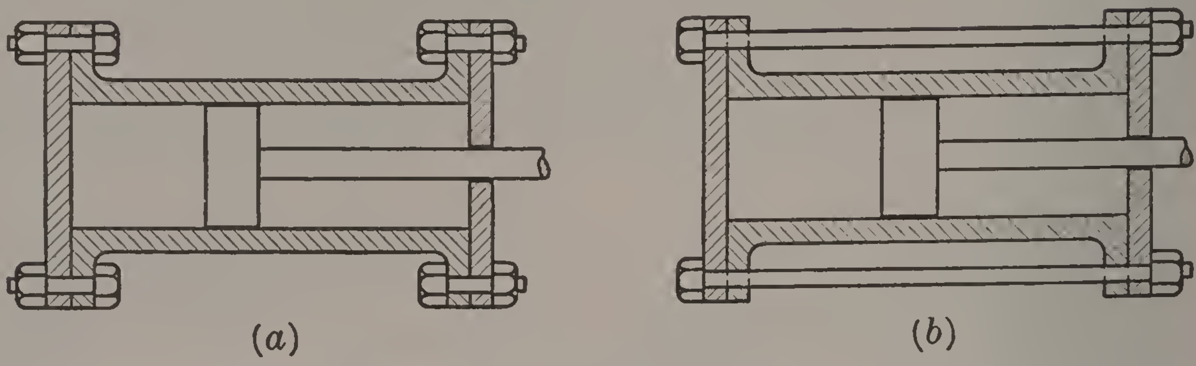

FIG. 225.—Energy resistance of bolts increased by making them longer.

118. Special Cases of Axial Energy Loads.—(*a*) *Sudden Load.*—A sudden load is a force that remains constant throughout the entire deformation, e, of the resisting member. Thus, if a sudden axial load P act on a bar the corresponding energy load, U, supplied to the bar (that is, the work done on the bar in causing the deformation e) is Pe.

Thus the expression for s in equation (214) becomes,

$$s=\sqrt{\frac{2PeE}{al}}, \qquad (217)$$

and since $\frac{e}{l}=\epsilon$, and $E\epsilon=s$, this expression may be written

$$s^2=2\frac{P}{a}s \quad \text{or} \quad s=2\frac{P}{a}. \qquad (218)$$

But if the load P were a static (gradually applied) load, $\frac{P}{a}$ would be the unit-stress developed. Therefore, within the proportional limit, the unit-stress caused by a sudden load is twice as great as that caused by the same load when gradually applied.

(*b*) *Energy Load Due to a Falling Body.*—If the energy load is delivered to a bar by a body of weight W falling through a height h before it comes in contact with the bar (see Fig. 229), the value of U in equation (214) is $W(h+e)$, where e is the total elongation of the bar. Hence,

$$s=\sqrt{\frac{2W(h+e)E}{al}}. \qquad (219)$$

But since e depends on s it will be convenient to express e in terms of s, and at the same time to express s in terms of the unit-stress, s_1, and deformation, e_1, that would be caused by the same weight if gradually applied. Thus, the above equation may be written:

$$s^2=\frac{2WE(h+e)}{al}.$$

But $\frac{W}{a}$ equals the unit-stress, s_1, due to the load W when gradually applied. Further, within the proportional limit, $s=E\epsilon$; also $e=\epsilon l$, and hence $\frac{E}{l}=\frac{s}{e}$. Therefore the above equation may be written:

$$s=2s_1\left(\frac{h}{e}+1\right).$$

Now the ratio of unit-stress to unit-deformation is assumed to be constant within the proportional limit regardless of the kind of load that causes the stress and deformation. Hence,

$$\frac{s}{\epsilon}=\frac{s_1}{\epsilon_1},$$

Also, $\frac{s}{e}=\frac{s_1}{e_1}$, since $e=\epsilon l$ and $e_1=\epsilon_1 l_1$,

in which s_1 is the unit-stress and e_1 the total deformation that would be caused by a static load W. Thus $e=\frac{se_1}{s_1}$ and hence

$$s=2s_1\left(\frac{hs_1}{se_1}+1\right).$$

Whence, the maximum stress is,

$$s=s_1+s_1\sqrt{1+\frac{2h}{e_1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (220)$$

in which s must not exceed the proportional limit. This equation shows that the stress due to an energy load caused by a falling weight may be greatly in excess of that caused by the same weight when gradually applied.

Similarly, since $\frac{s}{E}=\epsilon$ and $\frac{s_1}{E}=\epsilon_1$, the elongation e of the bar due to the falling weight is

$$e=e_1+e_1\sqrt{1+\frac{2h}{e_1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (221)$$

in which e_1 is the elongation of the bar that would be caused by the same weight if gradually applied.

119. Working Stress and Working Value of Energy.—If the equivalent-static-load method (see Art. 114) is used in making allowance for the effect of the impact of the load, the working stress may be the same with impact loading as with static loading. Specifications usually require, however, slightly better material (more uniform in quality and tougher) for use in structures subjected to impact, such as, ships, locomotives frames and bridges, than for structures subjected to loads that are approximately static loads, such as buildings.

If, on the other hand, the load is considered to be an energy load, the working stress, or rather the working value of the energy, should be based on the modulus of resilience $\left(\frac{1}{2}\frac{s_e^2}{E}\right)$ of the material, particularly if the maximum usable energy of the material is the elastic energy, that is, if the resisting member would be structurally damaged if stressed above the proportional limit (or yield point) of the material. But the toughness of the material should also be considered in order to insure reserve energy for resisting overloads and high localized stresses (see Art. 147 and 148 for further discussion).

A working value of energy should, of course, be considerably less than the modulus of resilience; the working value depends on the way the energy load is applied, on the service rendered by the member or machine, on the toughness of the material (its capacity to resist overloads, etc.), and on the form of the member with reference particularly to the way the energy is distributed throughout the member. In well designed machines (in which the energy load on any member is reduced to a minimum by the balancing of moving parts, by the use of springs, by adjusting the relative stiffness of the component parts and of the connected members, etc.), the working value of the energy may be made one-fifth to one-tenth of the modulus of resilience. It should be noted that the resilience varies as the square of the stress developed (assuming E to be constant as in the case of steel), and hence when the energy absorbed by the member is reduced a given amount the corresponding stress is reduced considerably less than this amount.

Table 5 should be helpful in selecting material for resisting energy loads and in selecting working values of energy to use in proportioning members subjected to energy loads.

TABLE 5

AVERAGE VALUES OF MODULUS OF RESILIENCE AND TOUGHNESS

Material	Tensile Proportional Limit (lb. per sq. in.) s_e	Tensile Ultimate Strength (lb. per sq. in.) s_u	Tensile Modulus of Elasticity (lb. per sq. in.) E	Ultimate Elongation Per Inch of Length (in.) ϵ_u	Tensile Modulus of Resilience (in.-lb. per cu. in.) $\frac{1}{2}\frac{s_e^2}{E}$	Toughness in Tension (Represented by area under Stress-strain Diagram) (in.-lb. per cu. in.)
Low carbon steel.....	30,000	60,000	30,000,000	0.35	15.0	15,700
Medium carbon steel..	45,000	85,000	30,000,000	0.25	33.7	16,300
High carbon steel.....	75,000	120,000	30,000,000	0.08	94.0	5,100
Special alloy steel (Heat treated).....	200,000	230,000	30,000,000	0.12	667.0	22,000
Gray cast iron.......	6,000	20,000	15,000,000	0.005	1.2	70
Malleable cast iron...	20,000	50,000	23,000,000	0.10	17.4	3,800
Rolled bronze........	40,000	65,000	14,000,000	0.30	57.2	15,500
Timber (Hickory)....	5,500*	10,000*	2,400,000*	......	6.32*	....

* In compression. The effectiveness of timber for resisting energy loads is greater than that indicated by the value of the resilience given in the table, due in part to the fact that timber may be stressed somewhat above its proportional limit without destroying its usefulness; that is, the properties of resilience and toughness are not sharply defined.

ILLUSTRATIVE PROBLEMS

Problem 219. Fig. 226 shows two bolts with square threads; they have the same dimensions except that one has the shank turned down to a diameter equal to that at the root of the threads for a length of 10 in. If both bolts

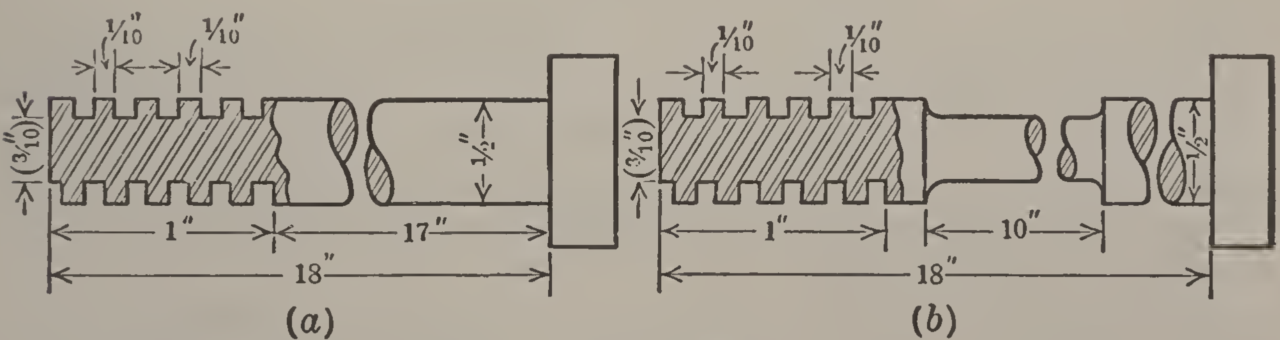

FIG. 226.

are made of soft steel with a tensile proportionality limit of 32,000 lb. per sq. in., find the axial static load and the axial energy load that each bolt will resist when stressed to the proportionality limit.

Solution.—Static Loads.—The static loads P that the two bolts can resist are equal since the least cross-sectional areas are equal. Thus,

$$P = as = \frac{\pi(0.3)^2}{4} 32{,}000 = 2{,}265 \text{ lb.}$$

Energy Loads.—The cross-sectional area, a_1, of each bolt at the root of the threads is 0.0706 sq. in. and a_2 of the shank is 0.1963 sq. in. Hence when the unit-stress s_1 in the first inch of the bolt (the threads are neglected) is 32,000 lb. per sq. in. the unit-stress s_2 in the shank of the bolt is only

$$\frac{0.0706}{0.1963} \times 32{,}000 = 11{,}520 \text{ lb. per. sq. in.}$$

Therefore, the energy load U that can be absorbed by the bolt in Fig. 226(a) when stressed to 32,000 lb. per sq. in. is

$$U = \frac{1}{2}\frac{s_1^2}{E}a_1l_1 + \frac{1}{2}\frac{s_2^2}{E}a_2l_2$$

$$= \frac{1}{2}\frac{(32{,}000)^2}{30{,}000{,}000}0.0706 \times 1 + \frac{1}{2}\frac{(11{,}520)^2}{30{,}000{,}000} \times 0.1963 \times 17$$

$$= 1.21 + 7.38 = 8.59 \text{ in.-lb. or } 0.715 \text{ ft.-lb.}$$

Likewise, the maximum energy load that can be applied to the bolt in Fig. 226(b) without causing a stress greater than the proportional limit is

$$U = \frac{1}{2}\frac{s_1^2}{E}a_1l_1 + \frac{1}{2}\frac{s_2^2}{E}a_2l_2$$

$$= \frac{1}{2}\frac{(32{,}000)^2}{30{,}000{,}000}0.0706 \times 11 + \frac{1}{2}\frac{(11{,}520)^2}{20{,}000{,}000}0.1963 \times 7$$

$$= 13.3 + 3.05 = 16.4 \text{ in.-lb. or } 1.36 \text{ ft.-lb.}$$

Therefore, a body weighing only 0.715 lb. falling a distance of only 1 ft. would supply enough energy to the bolt in Fig. 226(a) to stress it to the proportional limit *provided all the energy were absorbed by the bolt*, whereas a body weighing 2265 lb. would be required to cause the same stress if its weight were gradually applied to the bolt. Further, turning down the shank of the bolt as shown in Fig. 226(b) nearly doubles the energy resistance of the bolt without affecting its static resistance.

The results found in this problem further emphasize the importance of a uniform distribution of stress throughout a body that is required to resist energy loads. The results also indicate that energy loads would be extremely serious if the energy did not distribute itself throughout the whole structure or machine, or if it did not dissipate itself in the ways stated in Art. 115. These facts are further emphasized in the next problem.

Problem 220. In Fig. 227 is shown a punching machine. The flywheel A is made to rotate by means of friction discs (not shown); it supplies the energy required to punch a $\frac{7}{8}$-in. hole through a $\frac{5}{8}$-in. steel plate. The diameter of the screw at the roots of the threads is 4 in. and the length of the screw beneath the frame is 10 in.

(*a*) Find the maximum compressive unit-stress in the stem of the screw assuming that the load is a static axial load. (*b*) Find the maximum unit-stress, assuming that the load is an energy load and that the screw absorbs all the energy. (*c*) Find same as under (*b*), assuming that the machine frame absorbs three-fourths of the energy and the screw the other fourth.

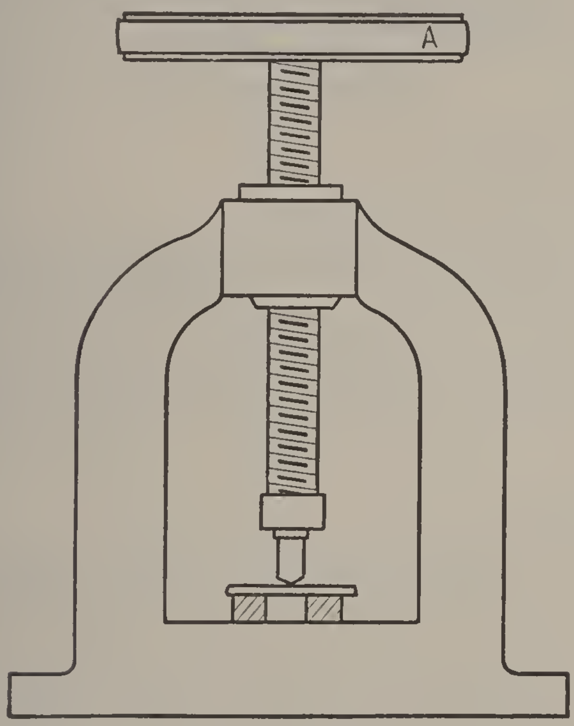

FIG. 227.

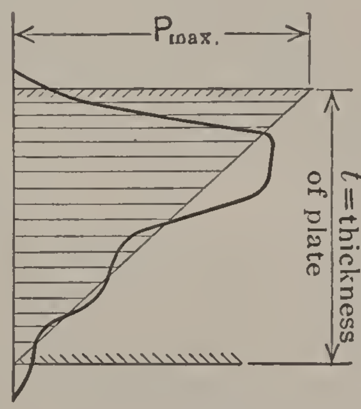

FIG. 228.

Solution.—(*a*) The maximum force P exerted on the punch, if the ultimate shearing unit-stress of the plate is 60,000 lb. per sq. in., is

$$P = \text{shearing area} \times 60{,}000$$

$$= \pi d \times t \times 60{,}000 = \pi \tfrac{7}{8} \times \tfrac{5}{8} \times 60{,}000$$

$$= 103{,}000 \text{ lb.}$$

And the unit-stress in the screw due to a static load of 103,000 lb. would be

$$s = \frac{P}{a} = \frac{103{,}000}{\pi(2)^2} = 8210 \text{ lb. per sq. in.}$$

(*b*) Tests show that the work diagram for the punching of steel plates is approximately that shown by the heavy line in Fig 228, and that the area under the work diagram is equal, approximately, to the area under a triangular diagram (Fig. 228) in which the maximum pressure is that required to develop a shearing unit-stress of 60,000 lb. per sq. in. in the steel in order to punch the hole. Hence, the work done in punching the hole as represented by the area under the triangular diagram is

$$U=\frac{P}{2}\times t=\frac{103{,}000}{2}\times\frac{5}{8}$$

$$=32{,}200 \text{ in.-lb.}$$

If all of this work were stored in the part of the screw between the plate and the machine frame and none in the machine frame, the unit-stress in the screw would be

$$s=\sqrt{\frac{2UE}{al}}=\sqrt{\frac{2\times 32{,}200\times 30{,}000{,}000}{12.56\times 10}}$$

$$=124{,}000 \text{ lb. per sq. in.}$$

provided that this value is less than the proportional limit.

(*c*) But the frame is not rigid, and hence probably a considerable part of the work done by the plate on the punch and screw is stored in the frame of the machine. If three-fourths of the work is used in deforming the frame then the value for U in the above expression is 8050 in.-lb. and the unit-stress becomes

$$s=62{,}000 \text{ lb. per sq. in.}$$

It is clear from the above results that a large part of the energy delivered to the screw is transmitted on to the machine frame. And, in general, the distribution of energy throughout the whole machine insures the safety of the machine; consideration of this fact is of great importance in the design of various types of machines and structures.

The results of this problem also suggest the desirability of a study of existing machines when designing members that are subjected to energy loads.

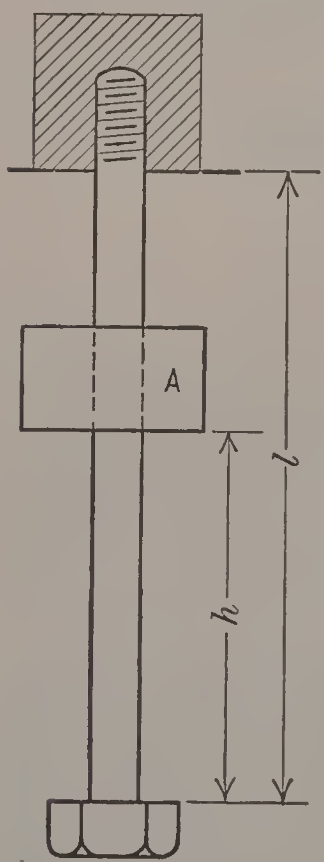

FIG. 229.

PROBLEMS

221. A body A, Fig. 229, having a weight of 20 lb. falls a height h of 4 ft. when it comes in contact with the end of a rod having a length l of 6 ft. (*a*) If the rod is made of soft steel having a proportional limit of 30,000 lb. per sq. in. what cross-sectional area should the rod have in order to prevent the unit-stress from exceeding the proportional limit? (*b*) If test specimens of the steel when tested had an ultimate strength of 60,000 lb. per sq. in. and an elongation of 30 per cent estimate the weight of A required to rupture a rod having a diameter of 0.25 in. (NOTE: All the energy of the falling weight is assumed to be absorbed by the rod.)

Ans. (*a*) $a=0.897$ sq. in.; (*b*) $W=684$ lb.

222. If the rod in Problem 223 were a stick of oak having a proportional limit of 3500 lb. per sq. in. and a modulus of elasticity of 1,800,000 lb. per sq. in., what should be the least cross-section of the stick to prevent the stress from exceeding the proportional limit? *Ans.* $a = 3.92$ sq. in.

223. The piston rods of the forging hammers in a certain steel plant frequently broke, and when they were finally replaced by a higher carbon steel the trouble ceased. If the proportional limit of the soft steel was 32,000 lb. per sq. in. and that of the higher carbon steel was 50,000 lb. per sq. in., how much more energy could the higher carbon steel rod absorb without being stressed above the proportional limit? Note: The breaking of the rods was probably due to the fact that the stress was repeated (repeated stress is discussed in the next chapter), but this was the secondary cause of the failure. (See footnote 2 of Art. 117.)

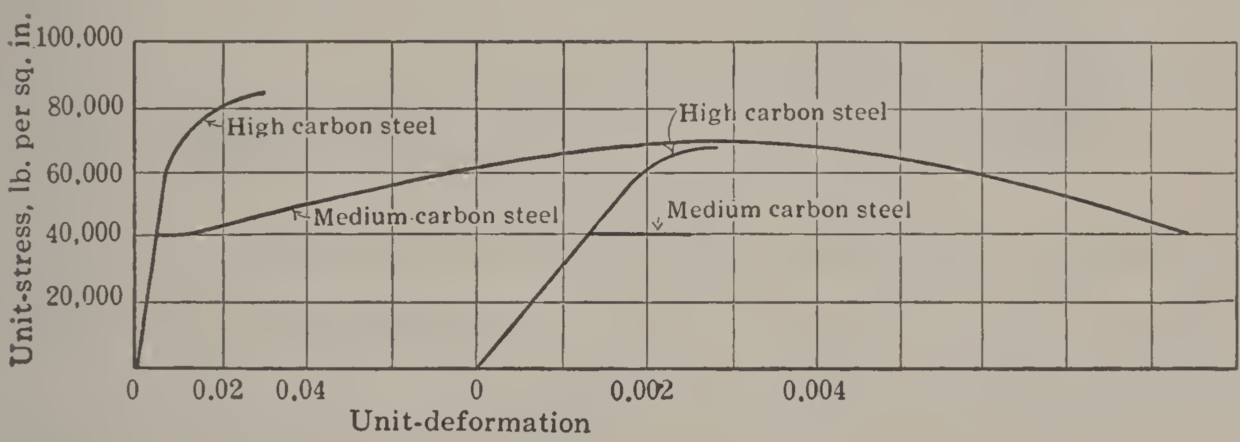

FIG. 230.—Stress-strain diagrams of two grades of steel.

224. Fig. 230 shows stress-strain diagrams for medium carbon steel, (about 0.30 per cent carbon) and for high carbon steel (about 0.80 per cent carbon). (*a*) If a bolt of medium carbon steel is stressed just to its proportional limit by an energy load of 25 foot-pounds, how large an energy load will a bolt of the same size of the 0.80 per cent carbon steel resist when stressed to its proportional limit? (*b*) What will be the ratio of the energy loads that will cause rupture of the two bolts? *Ans.* (*a*) 56.2 ft. lb.; (*b*) 6.5 approx.

225. In Fig. 231 are shown stress-strain diagrams for cast iron and for steel castings having about 0.20 per cent of carbon. Owing to the greater strength of steel the cross-sectional area of a steel casting for railway service (car couplers for example) is made only 0.4 as great as that of a cast-iron casting. What will be the relative resistance to rupture, when subjected to an axial energy load, of the steel coupler and the cast-iron coupler if the lengths are the same? Since the service of a car coupler is not destroyed by permanent deformations and since in railway service such deformations are very likely to occur, the ultimate rather than the elastic energy loads are the governing factors. *Ans.* 18.4 times as great.

226. A tension member in a certain machine is subjected to an energy load. If the member can be given a permanent deformation without destroying its usefulness and can be made of either cast iron or oak, which material should be used provided that the strength of the member is the only factor to be considered (make use of Fig. 231)? *Ans.* Oak.

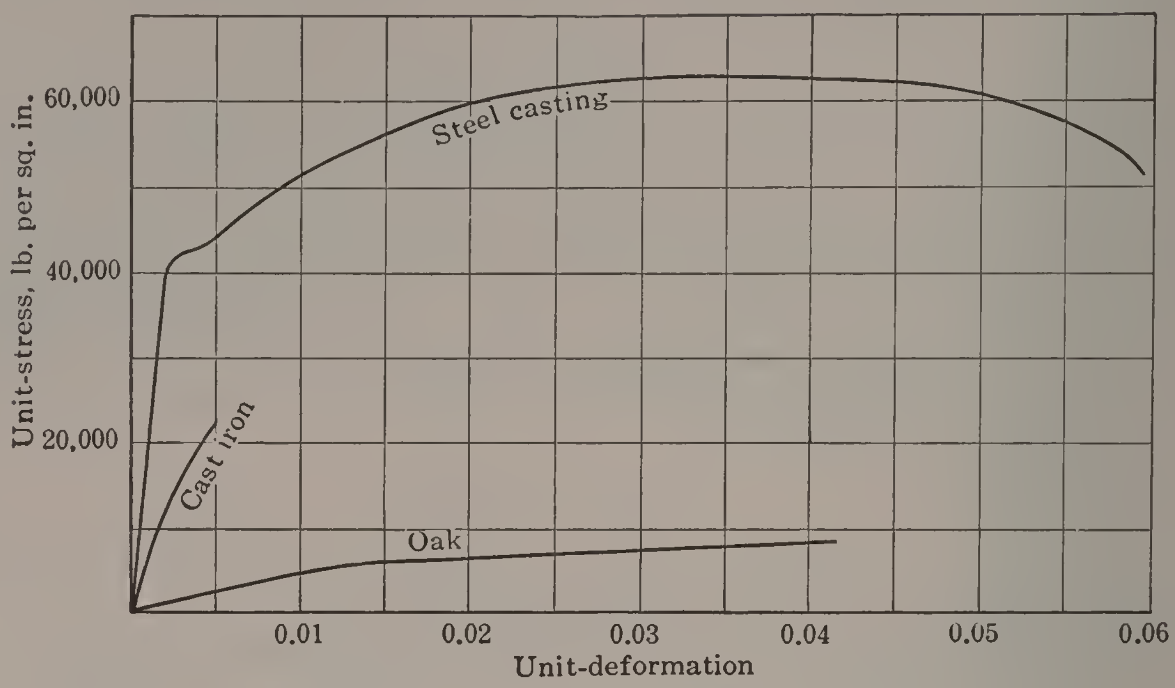

FIG. 231.—Stress-strain diagrams.

227. A bar 3 ft. in length and $\frac{1}{2}$ sq. in. in cross-sectional area is stressed to its proportional limit by an axial energy load U. If the diameter of the bar is turned down to one-half of its original value over one-fourth of its length, what is the value of the energy load that will stress the bar to its proportional limit?

120. Stresses in Beams Due to Energy Loads.—The amount of energy that a beam will absorb when stressed within the proportional limit of the material depends on the type of beam (cantilever, simple, fixed, etc.) and on the type of load (concentrated, distributed, etc.) in addition to the form and dimensions of the beam.

Simple Beam with Concentrated Load at Center.—Let it be required to find the maximum stress caused by an energy load U applied at the center of a simple beam, when the stress in the beam does not exceed the proportional limit of the material. Now the energy that can be stored in a beam is assumed to be the same when subjected to a gradually applied load as when subjected to an energy load due to a moving body. Thus the expression for the energy load U that can be absorbed by the beam when the stress does not exceed the proportional limit may be found as follows:

Let a simple beam having a span length of l feet be loaded with a gradually applied concentrated load at the mid-span, the final value of which is P pounds. The relation between the load P, and the deflection, Δ, of the beam is shown by the curve in Fig. 232(b). The energy load U that can be absorbed by the beam (which is equal to the work done on the beam as a gradually applied load increases from zero to the value P (represented by

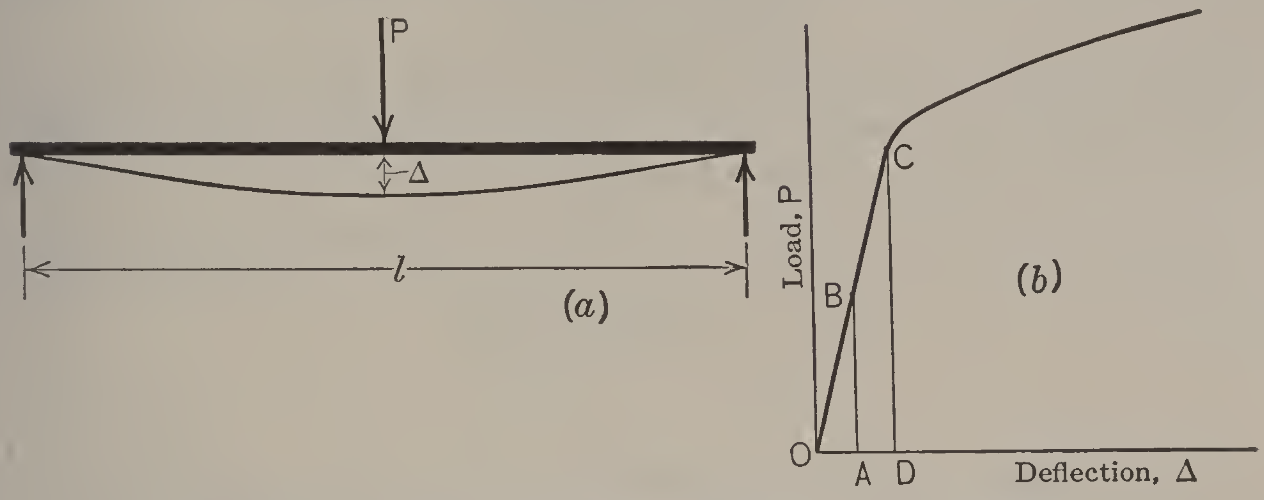

FIG. 232.—Energy resistance of beam; load at center.

AB in Fig. 232b) is represented by the area OBA, and may be found from the expression,

$$U = \tfrac{1}{2}P\Delta.$$

But from Art. 36 and 47 we have

$$\frac{Pl}{4} = \frac{sI}{c} \quad \text{and} \quad \Delta = \frac{1}{48}\frac{Pl^3}{EI}.$$

Therefore

$$U = \frac{1}{6}\frac{I}{c^2}\frac{s^2}{E}l. \qquad (222)$$

But $I = ak^2$ (Art. 163) in which k is the radius of gyration of the cross-sectional area with respect to the neutral axis. Hence

$$U = \frac{1}{6}\frac{k^2}{c^2}\frac{s^2}{E}al \quad \text{or} \quad s = \frac{c\sqrt{6}}{k}\cdot\sqrt{\frac{UE}{al}} \qquad (223)$$

If U is expressed in inch-pounds, E in pounds per square inch, a in square inches, and k, c, and l in inches, s will be expressed in pounds per square inch.

Since the numerical value of the ratio $\frac{k^2}{c^2}$ is always the same for similar shaped sections, the amount of energy that a horizontal rectangular beam will absorb when the short dimension is vertical will be the same as that absorbed when the long dimension is vertical.

PROBLEMS

228. Show that if the beam has a rectangular cross-section equation (223) reduces to

$$U=\frac{1}{18}\frac{s^2}{E}al=\frac{1}{9}\left(\frac{1}{2}\frac{s^2}{E}\right)al, \qquad (224)$$

which shows that the energy load which this beam can resist is only one-ninth as large as the energy load it could resist if it were used as a tension member and stressed to the same maximum value s.

229. Show that if the beam has a circular cross-section, equation (223) reduces to

$$U=\frac{1}{24}\frac{s^2}{E}al=\frac{1}{12}\left(\frac{1}{2}\frac{s^2}{E}\right)al. \qquad (225)$$

Simple Beam with Uniformly Distributed Load.—The relation between a gradually applied load and the deflection would be similar to that shown in Fig. 232(b). The final values of the load and maximum deflection are W and Δ, respectively. The deflection at any section is y and the load that acts through this distance is wdx (Fig. 233). Hence the energy load U that can be applied (which is equal to the work done on the beam) is

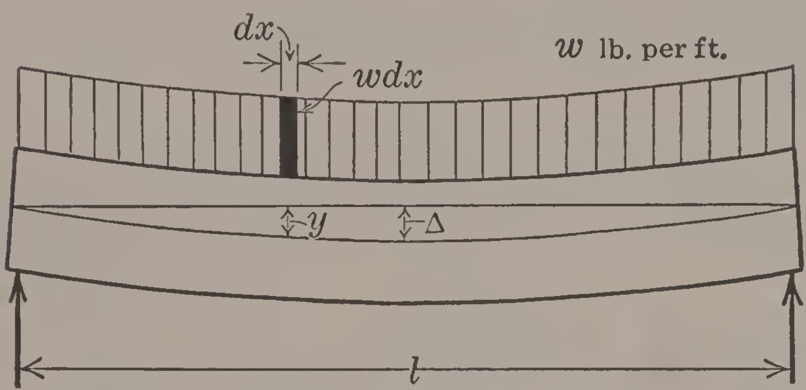

Fig. 233.—Energy resistance of beam; uniform load.

$$U=\int_0^l(\tfrac{1}{2}wdx\cdot y),$$

and by making use of the expression for y found in Art. 46, we have,

$$U=\frac{1}{2}\int_0^l wdx\frac{W}{EI}\left(\frac{l^3x}{24}-\frac{lx^3}{12}+\frac{x^4}{24}\right)$$

$$=\frac{1}{2}\frac{w^2}{EI}\left(\frac{l^5}{48}-\frac{l^5}{48}+\frac{l^5}{120}\right).$$

Hence

$$U=\frac{1}{240}\frac{w^2l^5}{EI} \qquad (226)$$

But w may be expressed in terms of the maximum unit-stress in the beam by means of the flexure formula, Art. 34; namely, $\frac{1}{8}wl^2=\frac{sI}{c}$. Therefore

$$U=\frac{1}{240}\left(\frac{8sI}{cl^2}\right)^2\frac{l^5}{EI}$$

$$=\frac{4}{15}\frac{I}{c^2}\frac{s^2}{E}\cdot l,$$

hence

$$U=\frac{4}{15}\frac{k^2}{c^2}\frac{s^2}{E}al. \qquad (227)$$

PROBLEMS

230. Show that if the beam has a rectangular cross-sectional area equation (227) reduces to

$$U=\frac{4}{45}\frac{s^2}{E}al=\frac{8}{45}\left(\frac{1}{2}\frac{s^2}{E}\right)al, \qquad (228)$$

which shows that the energy load which this beam can resist is only $\frac{8}{45}$ as large as the energy load it could resist if it were used as a tension member and subjected to the same maximum unit-stress.

231. Show that for a beam having a circular cross-sectional area equation 227 reduces to

$$U=\frac{1}{15}\frac{s^2}{E}al=\frac{2}{15}\left(\frac{1}{2}\frac{s^2}{E}\right)al. \qquad (229)$$

121. Effect of Form on Energy Resistance of Beams.—Equations (223) and (227) and the equations given in Problems 228 to 231 show that the material in a beam having a *constant* cross-section is inefficient in absorbing energy. For example, the expression in Problem 228 shows that a rectangular beam, when loaded at the mid-span with a concentrated load, can absorb only one-ninth as much energy as the same beam could absorb if all the material in the beam were stressed to the same degree.

The inefficient use of material in a beam having a constant cross-section for resisting an energy load arises from two causes: (1) since the bending moment is relatively small at sections near the supports (for simple beams) the unit-stress, even in the outer fibers at these sections, is necessarily small, and hence the material toward the ends of simple beams can absorb very little energy; and

(2) the material near the neutral plane throughout the length of the whole beam can develop only small stresses and hence can absorb only a small amount of energy per unit volume $\left(\frac{1}{2}\frac{s^2}{E}\right)$.

The first of these difficulties is largely overcome in leaf springs by making the leaves of gradually decreasing length (Fig. 234) so that the moment of inertia of the cross-sections decreases towards the supports approximately in proportion to the bending moment. Incidentally it is well to note that a leaf spring absorbs a considerable amount of energy due to friction between the leaves.

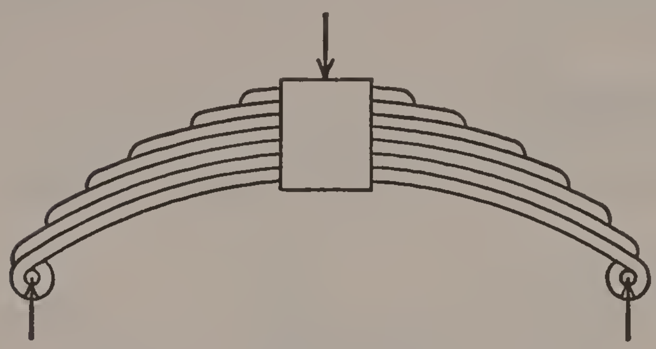

FIG. 234.

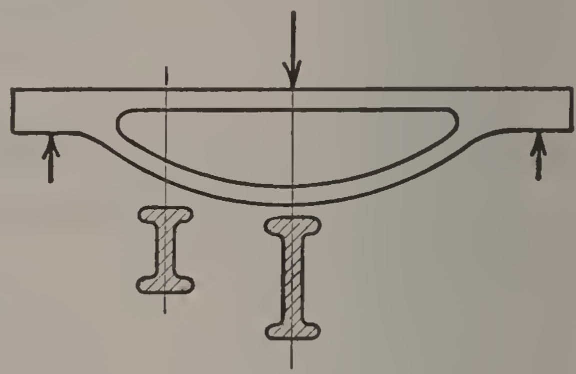

FIG. 235.

The second of the above difficulties may be overcome partly by placing as much of the material as far from the neutral surface as practicable, as for example, in the form of an I-section. If now a simple beam with an I-section is also made so that the cross-sectional area decreases towards the supports, as is sometimes done in forged axles, etc. (Fig. 235), the resistance of the beam to energy loads is very much greater than that of a beam having a constant cross-section that would have the same static strength as the beam with the variable cross-section.

It is clear, therefore, that when a beam is subjected to an energy load the distribution of the stress in the beam should be as uniform as possible throughout the beam so that the energies absorbed by all unit volumes $\left(\frac{1}{2}\frac{s^2}{E}\right)$ will be approximately equal.

Again, the energy absorbed by a beam as shown by equations (223) and (227) increases with the volume of the beam provided the other influencing factors discussed above are constant. Thus, if a simple beam has a constant cross-section the amount of energy it will absorb *increases* as the length of the span of the beam is increased, whereas the static strength *decreases* with an increase in the length of the span.

122. Special Cases of Energy Loads on Beams.—*Energy Load Due to a Falling Weight.*—It is convenient frequently to express the unit-stress caused by an energy load, due to a falling weight W, in terms of the unit-stress that would be caused by the force W when gradually applied. Similarly, the deflection caused by a falling weight W may be expressed in terms of the deflection that would be caused by a load W when gradually applied. This may be done by substituting in equation (223) or (227) the value of U for the beam in question similar to the method used in Art. 118; or it may be done as follows:

If a body having a weight W falls from a height h on a simply supported beam the maximum deflection, Δ, of the beam will be proportional to the maximum unit stress, s, developed and the ratio of s to Δ is assumed to be the same, within the proportional limit, as that of the unit-stress, s_1, to the deflection, Δ_1, caused by a static load equal to W, thus,

$$\frac{\Delta}{\Delta_1}=\frac{s}{s_1}, \qquad (230)$$

provided the proportional limit of the material is not exceeded.

Further, if Q is a static load that causes a deflection, Δ, equal to that caused by the energy load, the work done by Q, which is $\frac{1}{2}Q\Delta$, will be equal to the energy supplied or given up by the falling body, and since the assumption is here made that all the energy of the falling body, $W(h+\Delta)$, is absorbed in stressing the beam, then $\frac{1}{2}Q\Delta$ will be equal to the energy absorbed by the beam. Hence

$$W(h+\Delta)=\tfrac{1}{2}Q\Delta. \qquad (231)$$

But the static loads are proportional to the stresses they develop and hence from (1) we have

$$\frac{\Delta}{\Delta_1}=\frac{s}{s_1}=\frac{Q}{W}. \qquad (232)$$

By combining (231) and (232) there is found

$$s=s_1+s_1\sqrt{1+\frac{2h}{\Delta_1}} \quad \text{and} \quad \Delta=\Delta_1+\Delta_1\sqrt{1+\frac{2h}{\Delta_1}} \qquad (233)$$

in which s and Δ are the unit-stress and deflection, respectively, due to the falling body, and s_1 and Δ_1 are the stress and deflection, respectively, due to a static load equal to the weight of the falling body. Expressions for s_1 and Δ_1 are given in Chapters V and VI. Equations (233) show that a body which causes a relatively small stress and deflection when applied as a static load may cause a large stress and deflection if allowed to drop on the beam through a relatively short distance.

Sudden Load.—If the value of h in equation (233) is zero, that is, if the load W is a sudden load (see Art. 118 for definition of sudden load), then the values of s and Δ due to the sudden load are

$$s=2s_1 \quad \text{and} \quad \Delta=2\Delta_1. \qquad (234)$$

Therefore, a sudden load applied to a beam will cause twice the stress and twice the deflection that will be caused by the same load when gradually applied.

123. Deflection Due to Any Energy Load.—If an impact load U is delivered to a beam by any means other than that of a falling weight the deflection may be found from equation (233) as follows: After estimating the value of the energy U delivered to the beam, a value for W may be selected arbitrarily and a value of h that would be required to cause this weight W to deliver to the beam an amount of energy equal to U may be found. The unit-stress s_1 and the deflection Δ_1 that this weight would develop if applied gradually may also be found. Thus with values of h and Δ_1 known, equation (233) may be used to find the value of Δ. A value of s in equation (233) may also be found by a similar method, but equations (223), (227), etc., are preferable for this purpose.

ILLUSTRATIVE PROBLEM

Problem 232. The proportional limit for hickory may be taken at 3000 lb. per sq. in., and its modulus of elasticity at 1,500,000 lb. per sq. in. (*a*) Will a weight of 20 lb. falling 6 in. on the center of the span of a hickory beam 4 in. square cause a unit-stress above the proportional limit, if the beam has a span of 3 ft.? (Assume that the supports of the beam are rigid and that all the energy delivered by the falling weight is absorbed by the beam.) (*b*) How many inches will the beam deflect? (*c*) What should be the length of the span to make the stress equal 3000 lb. per sq. in.?

Solution.—(*a*) *First Method.*—The deflection of the beam will be neglected in comparison with 6 in. and hence the energy U delivered to the beam is

$$U=Wh=20\times6=120 \text{ in.-lb.}$$

The unit-stress caused by this energy load is (see Prob. 228).

$$s=\sqrt{\frac{18UE}{al}}=\sqrt{\frac{18\times120\times1{,}500{,}000}{4\times4\times36}}$$

$$=2370 \text{ lb. per sq. in.}$$

(*a*) *Second Method.*—If the deflection of the beam is not neglected the maximum unit-stress s developed in the beam is

$$s=s_1+s_1\sqrt{1+\frac{2h}{\Delta_1}}.$$

The unit-stress s_1 due to a central static load of 20 lb. is

$$s_1=\frac{Mc}{I}=\frac{1}{4}Pl\cdot\frac{c}{I}=\frac{1}{4}\times20\times36\times\frac{2}{\frac{1}{12}(4)^4}$$

$$=16.9 \text{ lb. per sq. in.}$$

The maximum deflection, Δ_1, of the beam caused by a central static load of 20 lb. is

$$\Delta_1=\frac{1}{48}\frac{Pl^3}{EI}=\frac{1}{48}\frac{20\times(36)^3}{1{,}500{,}000\times\frac{1}{12}(4)^4}$$

$$=0.000608 \text{ in.}$$

Hence,

$$s=16.9+16.9\sqrt{1+\frac{2\times6}{0.000608}}$$

$$=16.9+2365$$

$$=2382 \text{ lb. per sq. in.,}$$

and hence the error introduced in the first method of solution by neglecting the deflection of the beam is very small.

(*b*) The maximum deflection of the beam is

$$=\Delta_1+\Delta_1\sqrt{1+\frac{2h}{\Delta_1}}$$

$$=0.000608+0.000608\times140$$

$$=0.000608+0.0853$$

$$=0.0859 \text{ in.}$$

(*c*) Since the deflection may be neglected in determining the unit-stress, we have

$$s=\sqrt{\frac{18UE}{al}}=3000.$$

Hence

$$l=\frac{18UE}{a\times(3000)^2}=\frac{18\times120\times1{,}500{,}000}{16\times(3000)^2}$$

$$=22.5 \text{ in.}$$

PROBLEMS

233. Three beams, A, B and C, all of the same dimensions are arranged as shown in Fig. 236 similar to one type of draft gear. Each beam is 3 in. square and has a span of 12 in. (a) What static load applied at the mid-span will cause a maximum unit-stress of 20,000 lb. per sq. in.? (b) Will the beams when arranged as shown in Fig. 236 resist a greater static load than any one of the beams? (c) What is the weight of a body which, when allowed to fall a height h of 8 in. will cause a stress of 20,000 lb. per sq. in? (d) What stress would the weight in (c) cause if it were resisted by only one of the three beams? *Ans.* (a) 30,000 lb.; (b) no; (c) 30 lb.; (d) 34,500 lb. per sq. in.

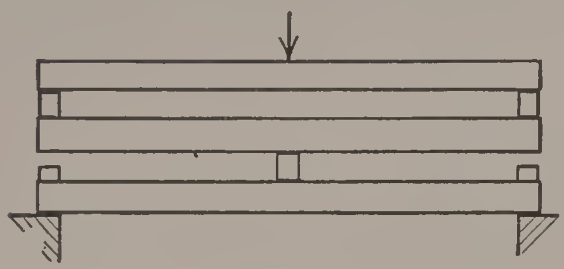

Fig. 236.

234. A steel bar 2 in. in diameter and 3 ft. long remains in a horizontal position as it falls a distance of 4 ft. and strikes rigid supports at its ends. If all the energy is absorbed by the beam what unit-stress is developed in the beam, assuming that the proportional limit is not exceeded.

235. A high carbon steel cylindrical shaft has a diameter of 4 in. It is used as a simple beam with a span of 6 ft. The proportional limit of the material is 60,000 lb. per sq. in. If a moving body strikes the beam at mid-span and two-thirds of the energy of the moving body is absorbed by the beam, how much energy does the body possess if it produces a unit-stress equal to one-half of the proportional limit? *Ans.* 1690 in.-lb.

CHAPTER XIV

REPEATED LOADS. FATIGUE OF METALS

124. Introduction.—A repeated load is a force that is applied many times to a member, causing stress in the material that continually varies, usually through some definite range. If a stress is developed in a member and is then released the member is said to have been subjected to a *cycle of stress.* Further, if a tensile stress has been developed and when released a compressive stress is developed and this stress is then released, the member is said to have been subjected to a *reversed cycle of stress* or, briefly, to a *reversal of stress*; the reversal of stress is said to be *complete* if the opposite stresses are of equal magnitudes. For example the piston rod of a steam engine that runs at 300 r.p.m. 10 hours per day 300 days per year for 10 years is subjected to 540,000,000 cycles of approximately complete reversals of axial stress; a car axle is subjected to about 50,000,000 complete reversals of bending stress in its normal "life"; a band saw in a normal service of about two months is subjected to about 10,000,000 cycles of stress, the stress in each cycle ranging from approximately zero to a maximum, etc.

Experience and experiments have shown clearly that the resistance of rolled or forged iron and steel (wrought ferrous metals) to repeated loads depends on very different action in the material than does the resistance to static or impact loading. For example, iron and steel will rupture when subjected to millions of reversals of stress not only when the calculated unit-stress in the material is less than the static ultimate strength of the material but even *when the calculated unit-stress is less than the static proportional limit.*

The failure of a material caused by repeated loads is a gradual or progressive failure. The failure seems to start at some point in the material at which the stress is much larger than the calculated stress, and this high localized stress develops a small crack which gradually spreads, as the load is repeated, until the whole member fails. The apparent loss of strength of a material due to repeated stress is frequently called *fatigue*; thus a fatigue failure is a failure caused by repeated loads. Failures of various machine members due to repeated stress have frequently occurred in service: Railway axles, steam turbine shafts and discs, crank shafts, piston rods, valve rods, springs in automobiles, etc., give much trouble in this respect.

Experimental Investigations.—In 1870 Wöhler published[1] the results of an extensive series of tests of various grades of iron and steel subjected to repeated direct tensile and compressive loads, to repeated bending loads, and to repeated torsional loads. The tests were carried out for the Prussian Railways during a period of about ten years. They are considered a classic in this field of investigation and until recently were the main source of our knowledge of resistance of material to repeated loads.

The number of cycles of stress to which the material was subjected in Wöhler's tests was usually less than 1,000,000, although a small number of specimens were stressed 10,000,000 times, a very few specimens were stressed 40,000,000 times, and one specimen 132,000,000 times. Although Wöhler's experiments gave reliable information for the machines and materials used at that time, the development of high speed machinery and of alloy and heat-treated steels have created a need for further experimental investigations. In recent years several important investigations[2] have been made that have added much to our knowledge of the subject.

125. Endurance Limit.—The *endurance limit of a material* is the maximum unit-stress that can be repeated, through a definite cycle or range of stress, an indefinitely large number of times without causing the material to rupture. As will be discussed later, the larger the cycle or range of stress is made the smaller the value of the endurance limit becomes; but when the term endurance limit is used without any limiting statement as to range of stress it will be understood to be the endurance limit with completely reversed cycles of stress.

[1] A summary of Wöhler's work is given in English in "The Testing of Materials of Construction" by Unwin.

[2] See Bulletins 124, 136 and 142 of the Engineering Experiment Station of the University of Illinois; a bibliography is given in Bulletin 124 and references to later work is given in Bulletins 136 and 142.

S–N Diagrams.—If several specimens are cut from the same bar of forged or of rolled steel [3] and are subjected to repeated complete reversals of stress (see Art. 156 for description of repeated-stress testing machines), it will be found that when a specimen is stressed nearly to the ultimate strength of the material in each cycle of stress the specimen will rupture after being subjected to a small number of cycles of stress; if a second specimen is tested in the same way but stressed slightly less than the first, a larger number of reversals of stress will be required to cause the specimen to rupture. Now, if a series of such experiments are carried out, the maximum unit-stress in any specimen being somewhat less than in the preceding specimen, the relation between the value of the completely reversed unit-stress, s, and the number of reversals, N, required to rupture the specimen will be found to be represented by a curve similar to that shown in Fig. 237 in which stresses are plotted as ordinates and numbers of reversals as abscissas.

The curve in Fig. 237 is called an s–N curve, and the ordinate to the s–N curve where the curve has become approximately horizontal is taken as a measure of the *endurance limit* of the material as defined above. Thus the endurance limit (s_r) obtained from the curve in Fig. 237 is approximately $\pm 19{,}000$ lb. per sq. in. That is, this material will rupture when subjected, in bending, to several million cycles of completely reversed stress if the maximum unit-stress in each cycle is slightly greater than 19,000 lb. per sq. in.

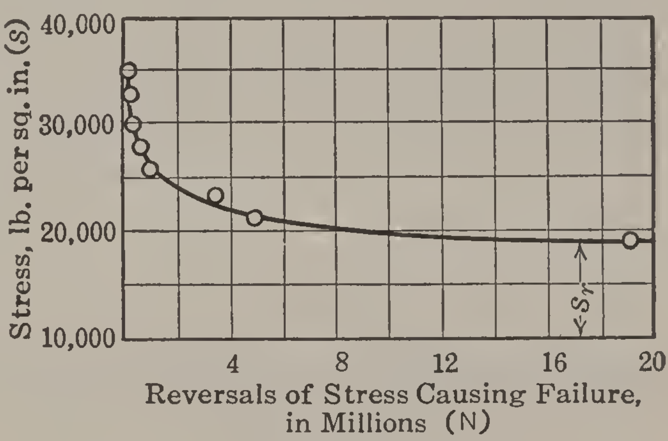

FIG. 237.— An s–N diagram for steel subjected to completely reversed cycles of bending stress.

Another way of obtaining an s–N curve and the endurance limit is to plot values of the logarithms of s and N (or the equivalent of this, namely, to plot values of s and N on logarithmic paper). When this is done the s–N curve (Fig. 238) is a straight sloping line until it changes its slope rather abruptly and becomes hori-

[3] Sufficient experimental data for castings of iron or steel and for non-ferrous metals from which to draw definite conclusions are not yet available.

zontal. The unit-stress at which the change of slope occurs is taken as the measure of the endurance limit (s_r).

Another convenient method of obtaining an s–N curve and the endurance limit is to plot the unit-stresses as ordinates and the logarithms of the numbers of reversals as abscissae, sometimes called semi-logarithmic plotting. The curves thus obtained are very much the same as those in Fig. 238. Thus, in Fig. 239, are shown s–N diagrams for several grades of steel plotted in two ways: In Fig. 239(a) values of s and N are plotted as Cartesian coordinates and in Fig. 239(b) is shown the semi-logarithmic plotting of the same values.

As shown in Fig. 238 and 239, wrought ferrous metals, if subjected to complete reversals of stress, will usually fail after resisting 1,000,000 to 5,000,000 cycles of stress, when the maximum stress in each cycle is slightly above the endurance limit. (Compare these values with the probable number of repetitions of stress in the "lifetime" of various members as given in Table 6 and determine if these members can safely be stressed above the endurance limit of the material.)

TABLE 6

Part of Structure or Machine	Approximate number of repetitions of stress in the "lifetime" of the structure or machine.
Railroad bridge, chord members	2,000,000
Elevated railroad structure, floor beams	40,000,000
Railroad rail, locomotive wheel loads	500,000
Railroad rail, car wheel loads	15,000,000
Airplane engine, crankshaft	18,000,000
Car axles	50,000,000
Automobile engine, crankshaft	120,000,000
Line shafting in shops	360,000,000
Steam engine, piston rods, connecting rods and crankshafts	1,000,000,000
Steam-turbine shafts	15,000,000,000
Steam-turbine blades	250,000,000,000

126. Localized Stress and Fatigue Failure.—When a *ductile* steel specimen is caused to fail by a gradually increasing (static)

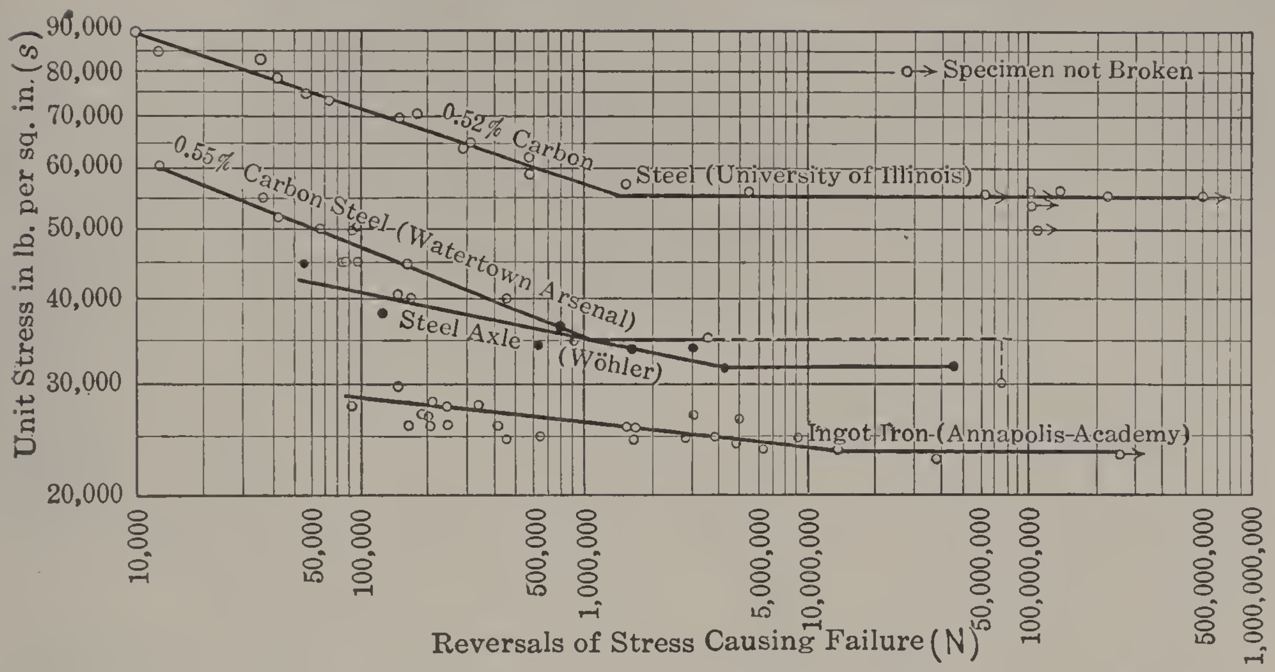

FIG. 238.—Logarithmic s–N diagrams for steel subjected to completely reversed cycles of bending stress.

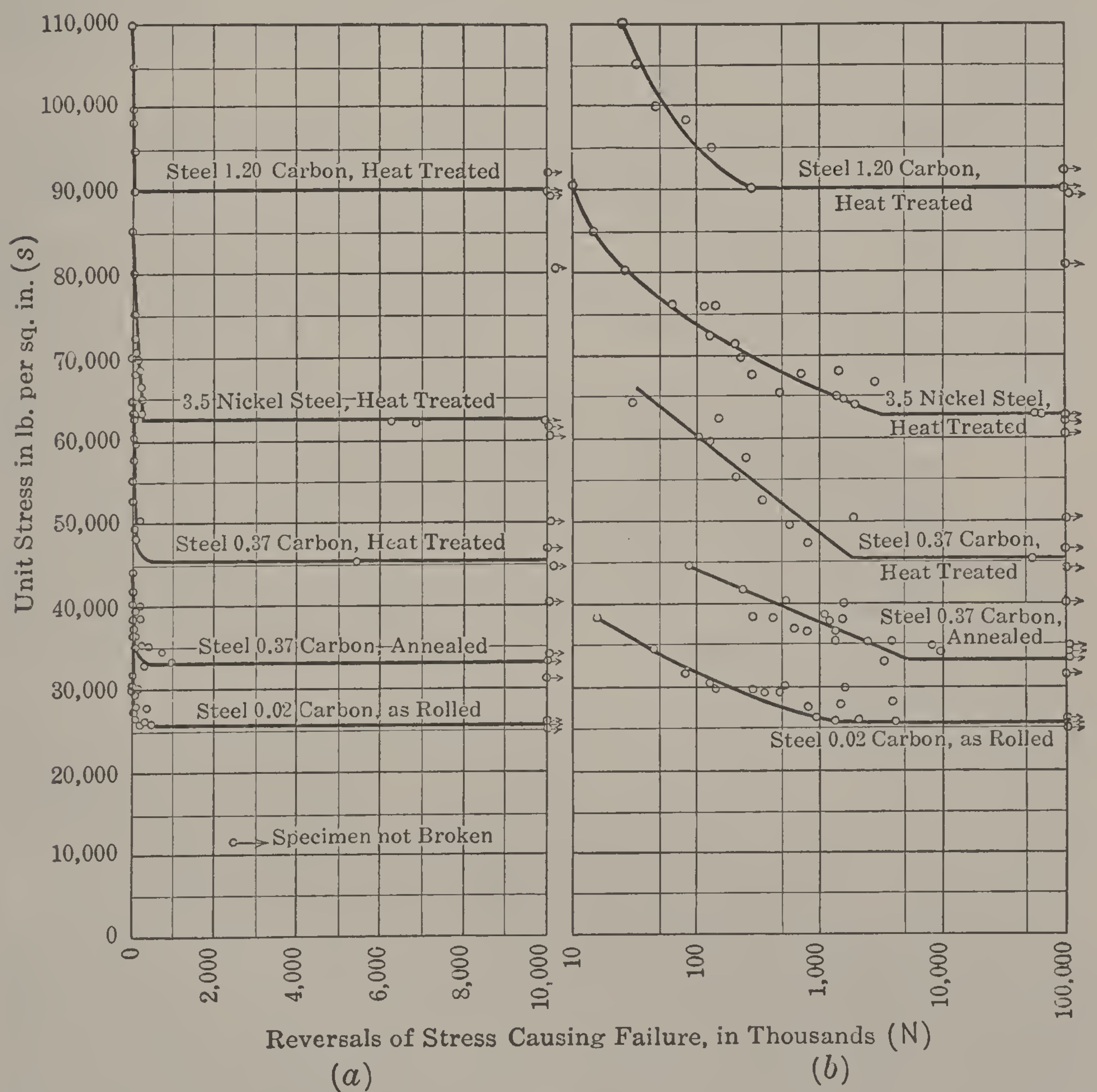

FIG. 239.—s–N diagrams for steel subjected to completely reversed cycles of bending stress; (a) Cartesian plotting; (b) semi-logarithmic plotting.

load there is visible evidence of structural damage (plastic deformation) and of the approaching of failure, considerably before the failure occurs. If a specimen of the same material, however, is caused to fail by a repeated load, there is no plastic deformation or other warning of the approaching of failure. Thus the *fatigue* failure of a ductile steel is similar to that of a *static* failure of a brittle material; the fractured area of a *ductile* steel specimen that is broken by repeated loads, usually presents a crystalline appearance similar to that of a coarse-grained brittle material, whereas the fractured area of the ductile steel specimen when broken by a static load presents a " silky " or " fibrous " appearance.

These observations suggested the *crystallization theory* of failure of steel due to repeated stress. It was thought that, in service, the repeated application of a load changed the steel from a fibrous ductile material to a crystalline brittle material. It is now known, however, that iron and steel are always crystalline and that the crystalline theory is entirely erroneous.

Localized Stress Theory.—The most satisfactory explanation of a fatigue failure is the *localized stress theory*, the main features of which may be explained as follows:

In determining the relation between stresses and static loads as expressed by the equations developed in the preceding chapters it was assumed

1. That the material was homogeneous; that is, there existed no discontinuities in the material and no abrupt changes in the properties of the material throughout the body.
2. That there was a definite regularity of stress distribution on any section of the member; that is, no discontinuities or abrupt changes occurred in the distribution of the stress over the section.

It is known, however, that these conditions never exist in structural members; metals for example are composed of crystalline grains whose strength and stiffness vary, and there are local concentrations of stress (localized stresses) at various portions of a member that may be much larger than the calculated values based on the assumed regularity of distribution. These localized stresses and irregularities in the stress distribution are due (1) to discontinuities in the material itself such is small flaws, fissures, nonmetallic inclusions, etc., at the edges of which high stress exist, and discontinuities in the properties of the material due to the variation in the strength and stiffness of the crystalline grains from those assumed for the material as a whole. Both of these causes of localized stress will be referred to as *internal discontinuities*, and (2) to *external discontinuities* of the material such as abrupt changes of sections, etc., where the stress distribution is, as a rule, radically different from that assumed.

Now when a ductile material is subjected to a static load, localized stresses that are considerably greater than the yield-point of the material may be developed (in fact the localized portion may even rupture) without seriously affecting the strength or deformation of the member as a whole, and hence the above assumptions are, in general, justified when the load is a static load (or is applied only a few times). But when the load is applied a very large number of times these localized stresses have a determining effect on the strength of the member (see Appendix III).

It seems clear that if a steel specimen *ruptures* when subjected to a repeated load, the stress in the material must have reached its ultimate strength even though the calculated stress may be less than the static proportional limit of the specimen. The rupture starts at a point of high localized stress or at a point where a weak crystal occurs, and gradually spreads until the whole member ruptures. The first experimental evidence of the gradual spread of the area of rupture was obtained by Ewing, Humphrey and Rosenhain.[4] They found that when the localized stress in certain crystals becomes sufficiently great the crystals yield by microscopic movement or sliding along their cleavage planes: These planes are called *slip planes* and their traces on a polished section of the member are seen under the microscope as dark lines, called *slip lines* or *slip bands* (see Fig. 240). For example, slip lines were detected in some of the crystals of a Swedish iron specimen when subjected to a few complete reversals of stress of $\pm 20{,}000$ lb. per sq. in., although the static yield-point and ultimate strength of the material were 31,600 and 52,800 lb. per sq. in., respectively; after more reversals of stress were applied additional slip lines appeared and the original ones broadened. Finally, various groups of slip lines united forming a visible crack which gradually extended until rupture occurred. (See Fig. 240*b*.)

[4] Philosophic Transactions Royal Society A, Vol. 200, p. 241, 1903.

However, not all slip lines develop into cracks, and there may be a development of slip lines and, possibly, of small cracks without resulting in failure of the member. In fact, fatigue failures frequently are not due to the development of slip bands but are caused by direct tearing apart at the points of high localized stress with subsequent spreading of the area of rupture, resulting in a failure that gives no warning of its approach. But, in any case, the cause of the failure is the localized stress due to internal or external discontinuities which are neglected in the usual formulas in mechanics of material as developed in the preceding chapters.

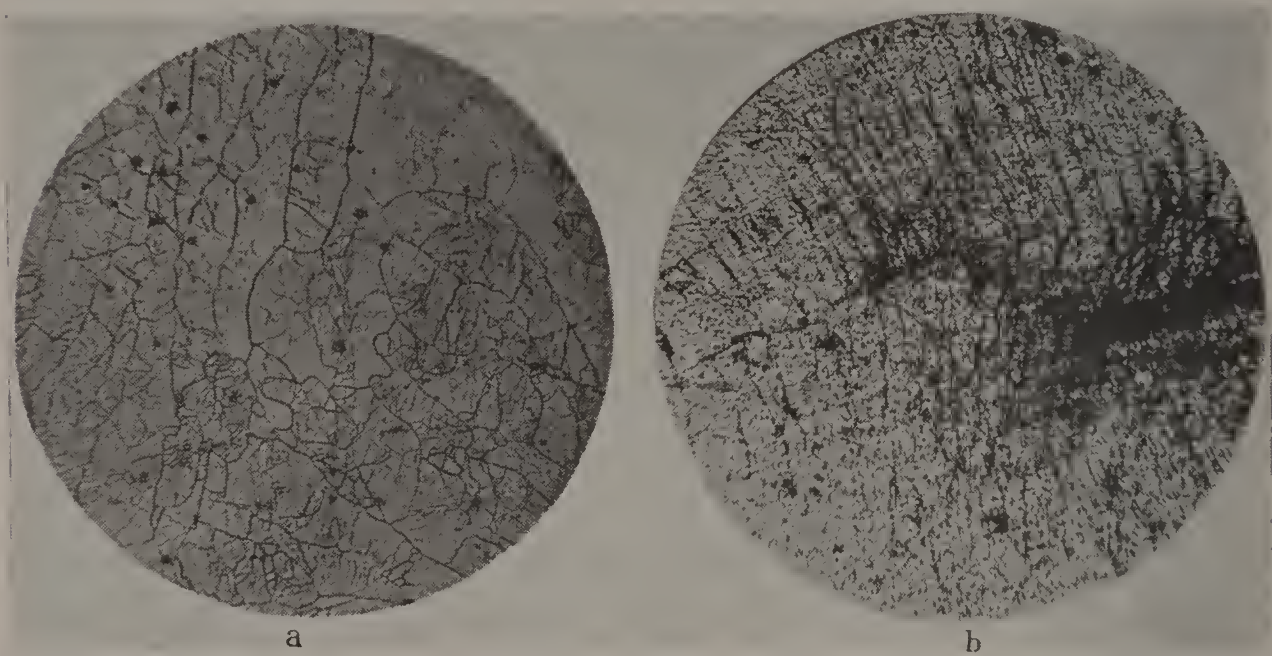

FIG. 240.—Photomicrographs of steel: (*a*) View before stressing, (*b*) view after application of several thousand reversals of large stress showing slip bands and crack. (Obtained by Prof. H. F. Moore.)

For example, in Fig. 241, is shown the section of a ruptured bolt that failed after being subjected to many repetitions of a tensile load. The failure started at the root of the thread where high localized stress occurred and spread inwardly. The dark portions of the cross-section show the area over which the crack spread and the light portion shows the area of rupture over which the material gave way suddenly.

127. Values of Endurance Limits with Completely Reversed Bending Stress.—Values of the endurance limits of several grades of steel when subjected to completely reversed bending stress are given in Table 7. Values of the static tensile proportional limits and of the tensile ultimate strengths of the steel are also given in the same table.

The values in Table 7 show that the endurance limit for complete reversals of stress may be considerably below the static proportional limit of the material. In other words, the *ultimate* resistance of steel to repeated loads is often considerably less than the *static elastic* resistance (proportional limit) of the material. Further, if a heat treatment greatly increases the static proportional limit it does not necessarily follow that the endurance limit is raised proportionally. In fact, tests show clearly that the endur-

FIG. 241.—View of area of rupture of bolt that failed due to repeated stress, showing evidence of progressive failure. (Obtained by Prof. H. F. Moore).

ance limit of steel bears a much more consant relation to the ultimate strength than to the elastic limit or proportional limit, and that for most rolled or forged steels the *endurance limit with completely reversed bending stress is approximately* 0.45 *of the static tensile ultimate strength* ($s_r = 0.45 s_u$) as may be seen by referring to Table 7.

128. Relation of Endurance Limit with Direct Axial Stress to Endurance Limit with Bending Stress.—The endurance limit, s_d, of a material when subjected to complete reversals of axial stress

TABLE 7

VALUES OF ENDURANCE LIMIT, ULTIMATE STRENGTH, PROPORTIONAL LIMIT, AND OF THE RATIO OF THE ENDURANCE LIMIT TO THE ULTIMATE STRENGTH AND TO THE PROPORTIONAL LIMIT FOR VARIOUS GRADES OF STEEL

(Specimens were subjected to completely reversed cycles of bending stress and were free from abrupt changes of section or surface scratches.)

Description of Steel		Endurance Limit, Lb. per Sq. In., s_r	Ultimate Tensile Strength, Lb. per Sq. In., s_u	Tensile Proportional Limit, Lb. per Sq. In., s_e	Ratio	
Composition of steel	Treatment				$\frac{s_r}{s_u}$	$\frac{s_r}{s_e}$
0.02% C (Ingot Iron)	As received (hot rolled)	25,000	42,400	16,100	0.58	1.56
0.21% C	As received (hot rolled)	34,000	70.000	39,900	.48	0.85
0.20% C	Cold drawn (surface polished after drawing)	41,000	80,800	55,200	.47	.72
0.30% C	Annealed*	30,000	69,900	35,000	.43	.86
0.57% C	Normalized*	33,000	71,900	34,500	.46	.96
0.37% C	Hardened† and Tempered	45,000	94,200	61,500	.48	.73
0.49% C	Hardened† and Tempered	48,000	96,900	67,700	.50	.71
0.77% C	Annealed	39,000	111,100	46,300	.35	.88
0.93% C (Spg. Steel)	Annealed‡	30,500	84,100	28,000	.36	1.09
0.93% C " "	Hardened and Tempered	56,000	97,100	84,800	.58	0.67
0.93% C " "	Hardened	98,000	188,300	106,500	.52	.92
1.20% C	Normalized‡	50,000	116,900	55,300	.43	.90
1.20% C	Hardened and Tempered	92,000	179,900	100,700	.51	.91
3.5% Nickel	Hardened and Tempered	63,000	118,000	86,400	.56	.73
3.5% Nickel	Annealed	54,000	101,600	60,800	.53	.89
Chrome-Nickel 3.3% Ni., 0.24% C, 0.87% Cr	Hardened and Tempered	68,000	138,700	122,600	.49	.57
Chrome-Nickel 3.3% Ni., 0.24% C, 0.87% Cr	Annealed	49,000	87,300	56,700	.56	.86
Nickel-Molybdenum	Hardened and Tempered	57,000	133,300	100,000	.43	.57
Chrome-Molybdenum	Hardened and Tempered	67,000	141,400	112,500	.48	.60
Chrome-Vanadium	Hardened and Tempered	67,000	146,600	114,000	.46	.59
Selico-Manganese (Spring steel)	Softer than is usual for springs	62.000	157,400	100.000	.40	.62

* Annealing consists in heating the steel above the upper critical temperature (about 800° C.) and then cooling the steel very slowly, as for example, cooling in the furnace after the fire is drawn. Normalizing differs from annealing in that the rate of cooling is somewhat faster, the material is usually drawn from the furnace and allowed to cool in the air.

† Hardening consists in heating the steel above its upper critical temperature and cooling it rapidly by quenching it in water or oil. Tempering consists in heating the material, after it has been quenched, to 450° C., more or less, to reduce the hardness slightly and to give the material a more uniform internal structure.

‡ This steel would seldom be used except in the hardened and tempered condition; it is included in the table mainly to show the effect of heat treatment.

(stress varying from a direct tensile to an equal direct compression stress) is found to be lower than the endurance limit, s_r, of the same material when subjected to complete reversals of bending stress.

A safe approximate value of s_d for rolled or forged iron and steel is

$$s_d = 0.6 s_r. \qquad (235)$$

The reason why s_d is less than s_r is due probably to the fact that the load always is applied with some eccentricity and hence the stress in the specimen is greater than the calculated stress; and also to the fact that in the bending-stress specimen only the outer fibers are subjected to the maximum stress whereas in the axial-stress specimen all the fibers are subjected to the same nominal stress.

129. Relation of Endurance Limits in Torsion and Bending.—The resistance of material to repeated *shearing* stress is of importance in some machine members such as torsional springs, shafts, etc., and although the number of repeated stress tests with reversed torsion is much less than with reversed bending, the tests are reasonably consistent in showing that the endurance limit of steel with complete reversals of torsional shearing stress, denoted by $(s_r)_s$, is 0.55 time the endurance limit with complete reversals of bending stress. Thus,

$$(s_r)_s = 0.55 s_r. \qquad (236)$$

130. Localized Stress Due to External Discontinuities.—(*a*) *Abrupt Change of Section.*—The results of several independent experimental investigations[5] have shown that high localized stresses occur at abrupt changes of section of a member when the member is resisting a load. Consequently, a repeated stress specimen having an abrupt change of section such as a square corner, keyway, screw thread, groove, etc., is found to fail when the calculated stress (found by use of the ordinary equations of mechanics of materials) is lower than the endurance limit of the material as found from specimens of the *same material* that are free from abrupt changes of sections. The ratio, then, of the endurance limit of the *material* to the endurance limit of the *specimen* with the abrupt change of section is a measure of the localized

[5] See Appendix III.

stress due to the abrupt change of section. For example, if the endurance limit of specimens free from abrupt changes of section is found to be 30,000 lb. per sq. in. and the endurance limit of specimens having given abrupt changes of section is found to be 15,000 lb. per sq. in., then the actual (localized) stress in the latter specimens is twice the calculated stress. Thus, repeated stress tests may be used to determine the value of localized stress. (Other methods of determining localized stresses are discussed in Appendix III.)

TABLE 8

THE REDUCTION IN THE ENDURANCE LIMIT OF A SPECIMEN OF STEEL SUBJECTED TO REVERSED BENDING DUE TO ABRUPT CHANGE IN SECTION; SPECIMENS WERE 0.4 INCH IN DIAMETER.

Form of Section	Reduction in Endurance Limit in Per Cent
Groove with 10-in. radius	0
Groove with 1-in radius	5
Groove with $\frac{1}{4}$-in. radius	10
Shoulder or notch with small fillet	25
Square shoulder	50
Sharp V-notch	65

It is very clear from the results in Table 7 that, in designing members that are subjected to repeated loads, sharp corners should be avoided as much as possible, and experience has emphasized the truth of this statement. For example, the gears, crank shafts, and connecting rods of the Liberty airplane engine, at first, frequently failed from repeated stress; the failure in the gears started at the sharp corner at the bottom of the teeth, of the crank shafts at the corner of the keyways, and of the connecting rods at the sharp corner where the connecting rod bolt-head fitted on the assembly. Further, the British found that, when a failure of an airplane crankshaft was due to a sharp corner, failure after failure would occur in engines of the same type in approximately the same number of running hours. The trouble in the Liberty motor parts finally was practically eliminated in all parts by adopting a small fillet of about $\frac{1}{32}$ in. radius at each corner.

(*b*) *Surface Finish.*—Experiments [6] have shown clearly that the surface finish of a member has an appreciable effect on its endurance limit, the smoother finish giving the higher value for the endurance limit. Since scratches, tool marks, indentations due to cold pressing, etc., may be regarded as more or less abrupt changes of section, giving rise to localized stress, this effect of surface finish is to be expected.

The results of experiments [6] show that a specimen having a surface produced by turning a finishing cut in a lathe may have an endurance limit as much as 10 per cent less than a specimen having a surface produced by fine grinding, and somewhat rougher turning may cause a reduction in the endurance limit of 20 per cent or more. Although fine grinding does not produce as good results as does polishing, it would probably be considered satisfactory for many commercial machine members.

It is clear, therefore, that surface scratches due, for example, to grit in bearings, indentations due to hammer blows or high bearing pressure, tool and die marks in cold pressed work, etc., must be taken into account in estimating the resistance of a member to repeated loads.

For example, repeated stress failures of the cold pressed water jackets on certain airplane engines have been found to be due to die marks. And, the observation of fractures that have occurred in service shows that slight surface scratches appear to determine the position of a fracture in steel subjected to reversals of stress.

131. Working Stress with Repeated Loads.—The endurance limit of steel subjected to millions of cycles of stress is really the fatigue *ultimate* strength of the material since if the unit-stress developed in the material is greater than the endurance limit the material will rupture. Further, there is no unit-stress below the endurance limit at which the beginning of inelastic action or structural damage can be detected, whereas with static loading the beginning of structural damage can be detected at stresses much below the static ultimate strength of the material since the proportional limit, yield point, etc., indicate the beginning of inelastic action or structural damage caused by static loads. It seems clear, therefore, that the working stress for steel members subjected to repeated stress should be based on the endurance limit of the material.

If, however, the endurance limit of the steel is not known

[6] Bulletin 124, Engineering Experiment Station, University of Illinois.

(see Art. 156 for machines and methods of testing) an approximate value of the endurance limit, may be found from the static ultimate strength of the steel since tests have shown, as stated in Art. 127, that the endurance limit for completely reversed cycles of stress may be taken with reasonable accuracy as 0.45 (or roughly 0.5) of the static tensile ultimate strength of the material.

Further, if the stress is not completely reversed, an approximate value of the endurance limit for a given range of stress may be obtained from the endurance limit with completely reversed stress by means of equation (240) or (241) in Art. 132.

Now it is important to note that the values of the endurance limits found from the usual laboratory tests are obtained with a specimen having a very gradual change in section and having a polished surface finish, and hence the endurance limit of the *specimen* may be considered to be the endurance limit of the *material*, the internal discontinuities being assumed constant for a given material. Again, it is only when these conditions exist that the endurance limit may be assumed to be 0.45 (or roughly 0.5) of the static tensile ultimate strength of the material.

Need for Margin of Safety.—As discussed in Art. 7, three of the reasons for selecting a working stress less than the maximum usable strength (yield point, etc.) of the material, in the case of *static* loading, are (1) that the actual loads to which the member will be subjected are seldom known with certainty, (2) that the actual stresses in a member, even if the member were subjected to *known* loads, may be considerably greater than those calculated by the usual formulas (as developed in the preceding chapters), since these formulas do not take account of the localized stresses due to various causes (see Art. 138 and 143) and (3) that the properties of the material vary, due to variation in the quality of the material, from those obtained from the tests of sample specimens.

Now the uncertainty of the loads in the case of *repeated* loading is probably about the same as in static loading, and about the same allowance in selecting the working stress should be made for the uncertainty of the load with both types of loading. Further, the uncertainties in the calculated stresses in a member due to the presence of localized stress are probably no greater with repeated loading than with static loading, *but* the effect of localized stresses have a *very much greater effect* on the resistance of the material to repeated loads than on the resistance to static loads.

For, as discussed in Art. 126 and 143, and in Appendix III, when localized stresses that are considerably in excess of the calculated stresses occur in a steel member that is subjected to *static* loads (and such localized stresses always exist) the material at the point where a localized stress occurs will yield, since the steel is fairly ductile, and the stress at this point will be relieved, thereby transmitting some of the excess stress to the surrounding, less-stressed material, and the resistance of the member *as a whole* will not be seriously affected by this action; whereas, if the same member were subjected to *repeated* loads, experience and experiments show that a minute crack would probably start at the point of high localized stress and would gradually spread, as the loads were repeated, until the whole member would rupture.

Now, since localized stresses are so important in repeated stress members and since experimental data are now available by means of which a reasonably close estimate of the intensity of the localized stress in a member can be made, *when the localized stress is due to external discontinuities* such as abrupt changes of section, surface finish, etc., it seems unnecessarily indefinite to make allowance for localized stress in repeated-stress members by selecting, arbitrarily, low working stresses.

Method of Calculating Maximum Stress.—The method of obtaining the maximum (localized) unit-stress due to external discontinuities is as follows: Calculate, according to the usual formulas used for static loading, the maximum unit-stress that the loads cause, and then multiply this unit-stress by a factor to obtain the intensity of the localized stress, the factor depending on the form of the member. The value of the factor is obtained from the results of repeated-stress experiments and from other methods discussed in Appendix III. Approximate values for these factors for steel members having various external discontinuities are given in Table 9. The value of the factor is unity if the member has a very gradual change of section and a very smooth surface, that is, if the member has no external discontinuities.

Selection of Working Stress.—Now, as explained in Art. 126, localized stresses arise from *internal* discontinuities as well as from *external* discontinuities, and the localized stresses due to the internal discontinuities, such as the stresses that occur at the edges of the minute blowholes, pipes, etc., and at points of high-bearing stress of two or more crystals due to a disadvantageous arrangement of the crystals, etc., cause a specimen even when free from external discontinuities to rupture, when subjected to repeated stress, at a calculated unit-stress less than the static elastic limit of the material. In other words, the unit-stress *on some minute area* in the specimen exceeds the ultimate strength of the material at that point even when the maximum *calculated* stress in the material is considerably less than the static elastic limit of the material.

TABLE 9

MAXIMUM LOCALIZED STRESSES DUE TO VARIOUS EXTERNAL DISCONTINUITIES

Concentric Groove around Cylindrical Shaft: Ratio of radius of groove or fillet to diameter of shaft.	Ratio of maximum localized unit-stress to computed maximum unit-stress in shaft.
0.1	2.0
0.5	1.6
1.0	1.2
2.0	1.1
Special Types of Abrupt Changes of Section and of Surface Finish:	
Square corner	2.0
Sharp V-thread	3.0
Whitworth thread	2.0
U. S. Standard thread	2.5
Surface finish produced by lathe tool; (Scratches due to grit in bearings)	1.2
Surface finish produced by a grinding wheel	1.05

But, there is no reliable way of detecting or of measuring the effects of internal discontinuities as can be done for the external discontinuities. Therefore, an allowance is made for the effect of these internal conditions by selecting a working stress as some proportion of the endurance limit found from the usual repeated-stress tests. Experiments indicate that specimens taken from the same bar of steel and made and tested as nearly alike as possible rupture, when subjected to repeated loads, at calculated unit-stresses that may vary considerably, and the endurance limits of specimens taken from different bars of a shipment of steel of presumably the same quality may vary greatly.

Thus, even when the limiting values of the repeated loads are known (as for example in the case of a spring provided with tops to prevent excessive deflection), and even when due allowance has been made, as discussed above, for the effect of the external discontinuities of the member, it is recommended that the working stress should not be greater than one-half the endurance limit in order to make allowance for the effect of internal discontinuities, and that the working stress should not be greater than one-third of the endurance limit in order to make allowance for the uncertainties in the assumed loads, in addition to the uncertainties in the internal conditions of the metal. However, in using a working stress equal to one-third of the endurance limit it is assumed that the maximum calculated stress in the material is the *localized* stress arising from eternal discontinuities as discussed above, and not merely a nominal stress found from the ordinary equations of mechanics of materials.

It is assumed throughout this chapter that the repeated loads are not applied with impact. If a member is subjected to repeated impact loads some additional allowance must be made for the effect of the impact, depending on the conditions of the problem.

Again, some machine members in service are subjected to repeated stresses greater than the endurance limit and are not expected to resist these stresses indefinitely as for example, band saws for wood, wire rope running on sheaves, ball bearings, gun barrels, etc.

ILLUSTRATIVE PROBLEMS

Problem 236. The crank pin (Fig. 242) has a diameter, d, of 5 in. and a length of 6.25 in., these dimensions being determined from the allowable bearing pressure. The shaft is made of the normalized 0.57 per cent carbon steel listed in Table 7. The radius of the fillet at section AA is $\frac{1}{32}$ in. The connecting rod exerts a total pressure of 25,000 lb. on the pin. If the pressure is assumed to be uniformly distributed, will the pin probably fail, due to repeated stress, in a service of ten years? If not, what is the ratio of the localized stress to the endurance limit? The speed of the shaft is 120 r.p.m., the engine runs ten hours per day and three hundred days per year.

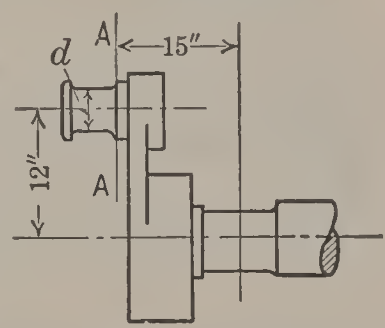

Fig. 242.

Solution.—The crank pin would be subjected to about 200,000,000 complete reversals of bending stress and hence would fail if the unit-stress in the pin exceeds the endurance limit of the material. The endurance limit of this material, as given in Table 7, is 33,000 lb. per sq. in.

The maximum unit-stress in the pin is found as follows: The nominal unit-stress as given by the flexure formula is

$$s=\frac{Mc}{I}=\frac{25{,}000\times 3.125\times 2.5}{\frac{1}{64}\pi(5)^4}=6360 \text{ lb. per sq. in.}$$

and by making use of Tables 8 and 9 as a guide, the maximum unit-stress at the root of the fillet will be estimated to be twice the calculated value. Thus,

$$\text{maximum } s=6360\times 2=12{,}720 \text{ lb. per sq. in.}$$

which is less than the endurance limit. The ratio, f, of the localized stress to the endurance limit, then, is

$$f=\frac{12{,}720}{33{,}000}=\frac{1}{2.6},$$

and hence the stress in the crank pin is slightly greater than the permissible working stress recommended.

Problem 237. A piston rod (Fig. 243) is to be subjected to a maximum load, P, of 60,000 lb. The engine runs at a speed of 150 r.p.m. and will be assumed to be in service six hours per day, three hundred days per year for at least ten years. The rod is made of 0.4 per cent carbon steel heat treated to give an endurance limit with completely reversed bending stress of 46,700 lb. per sq. in. (compare with Table 7). The rod has sharp corners at B, and the diameter of the rod at the roots of the threads may be assumed to be 0.85 d. What should be the diameter, d, of the rod?

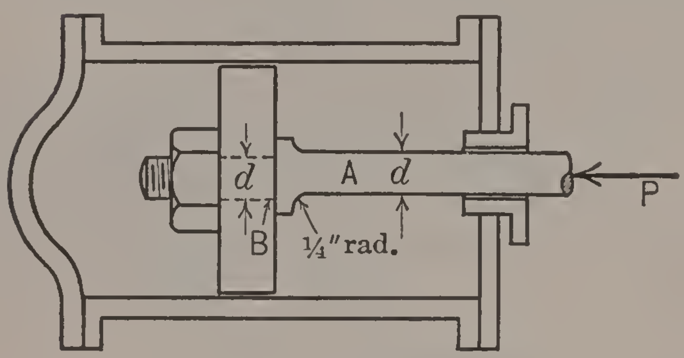

FIG. 243.

Solution.—The endurance limit for the material (see Art. 128) is,

$$s_d=46{,}700\times 0.6=28{,}000 \text{ lb. per sq. in.}$$

If the working unit-stress (s_w) is taken as one-third of the endurance limit, then,

$$s_w=\frac{28{,}000}{3}=9330 \text{ lb. per sq. in.}$$

Now the average or nominal unit-stress on the cross-section at the roots of the thread is

$$s_{av.}=\frac{P}{a}=\frac{60{,}000}{\frac{\pi(0.85d)^2}{4}},$$

and the maximum (localized) unit-stress at the sharp corner, B, or at the root of the threads, will be two or three times the average unit-stress. A value of

3 will be used since it is likely that the stress would be far from constant over the area due to eccentricity of loading even if there were no abrupt changes of section. Therefore,

$$s_{max.}=3\frac{P}{a}.$$

Hence

$$9330=3\frac{60,000\times 4}{\pi(0.85d)^2}$$

$$d=5.83 \text{ in.}$$

PROBLEMS

238. The rods which operate the slide valves of engines frequently break. One method of construction is shown in Fig. 244. If the threads on the rod are U. S. standard threads, estimate a safe working stress for the rod if the material is hot-rolled low-carbon steel (about 0.20 per cent carbon, Table 7); also find the corresponding working value of the load P if the cross-sectional area of the bar is 2 sq. in.

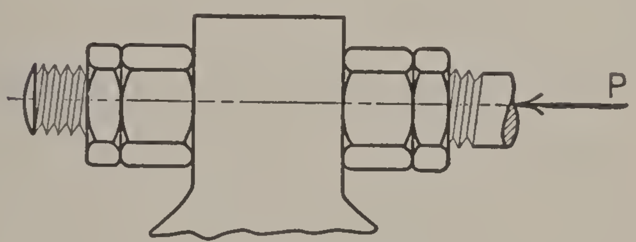

FIG. 244.

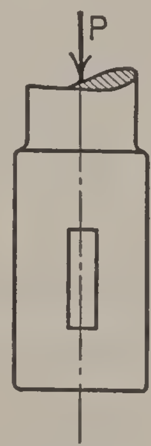

FIG. 245.

239. Fatigue failures frequently occur at the corners of cotter holes. If a bar having a rectangular cotter hole (Fig. 245) with square corners is subjected to repeated direct stress varying from zero to a maximum, and the bar is made of 0.40 per cent carbon steel having a tensile ultimate strength of 80,000 lb. per sq. in., what working unit-stress should be used in the design of the bar? What working axial load, P, should be applied to the bar if the cross-sectional area of the bar is 4 sq. in.?

240. A circular shaft having a constant diameter of 2 in. is to be subjected to several million complete reversals of bending stress. The shaft is turned with a lathe tool. What is the maximum bending moment that should be applied to the shaft, if the shaft is made of about 0.50 carbon steel, hardened and tempered (see Table 7).

241. In Fig. 246 are shown two forms of crank pins. The dimensions l and d of the crank pins are equal and the pins are made of the same grade of steel. Compare the maximum repeated pressures the two pins can resist without breaking due to repeated complete reversals of bending stress.

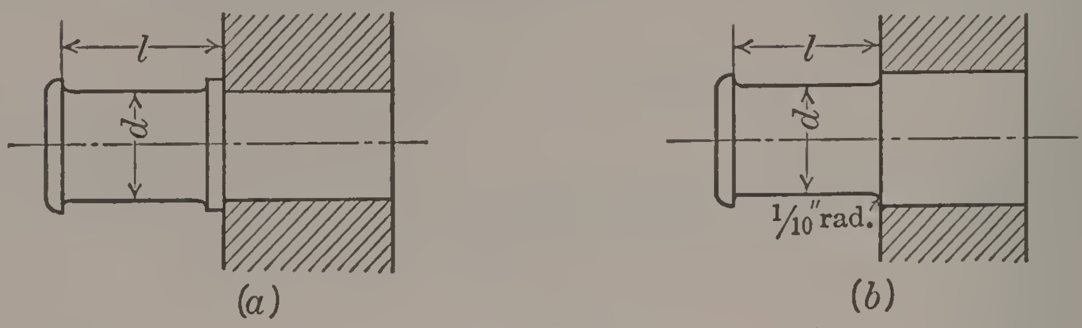

Fig. 246.

132. Effect of Range of Stress.—The investigations of Wöhler and the supplementary work of Bauschinger showed that the number of cycles of stress required to cause rupture depends on the range of stress. For example, if the unit-stress in each cycle varies from a tensile stress of 30,000 lb. per sq. in. to a compressive stress of 30,000 lb. per sq. in. (a range of stress of 60,000 lb. per sq. in.) fewer repetitions are necessary to cause rupture than if the unit-stress in each cycle varies from zero to 30,000 lb. per sq. in. although the maximum unit-stress is the same for each of these ranges of stress.

Further, the maximum or limiting unit-stress that can be repeated an indefinitely large number of times (the endurance limit) depends on the lower value of the stress in the cycle of stress and hence indirectly on the range of stress in the cycle. A formula based mainly on the results of Wöhler's experiments, that expresses the relation between the endurance limit of a material and the range of stress was developed independently by Goodman [7] and by J. B. Johnson,[8] and will here be referred to as the Goodman-Johnson formula. It may be derived as follows:

Goodman-Johnson Formula.—In Fig. 247 the minimum unit-stresses ($s_{min.}$) or lower limits of the various ranges of stress (Δ^5) which, in Wöhler's experiments, resulted in failure after the application of about 4,000,000 cycles of stress were plotted as ordinates, the abscissas being selected arbitrarily so that a straight line (EOC) would connect the ends of the ordinates. It was then found that if the corresponding maximum unit-stresses (s_{max}). or upper limits of the ranges of stress were plotted as ordinates the ends of the ordinates fell approximately on the straight line CAD

[7] Mechanics Applied to Engineering, p. 634.
[8] Johnson's Material of Construction, 5th Edition, p. 781.

(Fig. 247) where OA is equal to one-half of the static ultimate strength of the material ($\frac{1}{2}s_u$), and MD and ME are equal to one-third of the static ultimate strength ($\frac{1}{3}s_u$). Thus, for complete reversals of stress the endurance limit was found by Wöhler to be equal to about $\frac{1}{3}s_u$,[9] and for a range of stress with the mini-

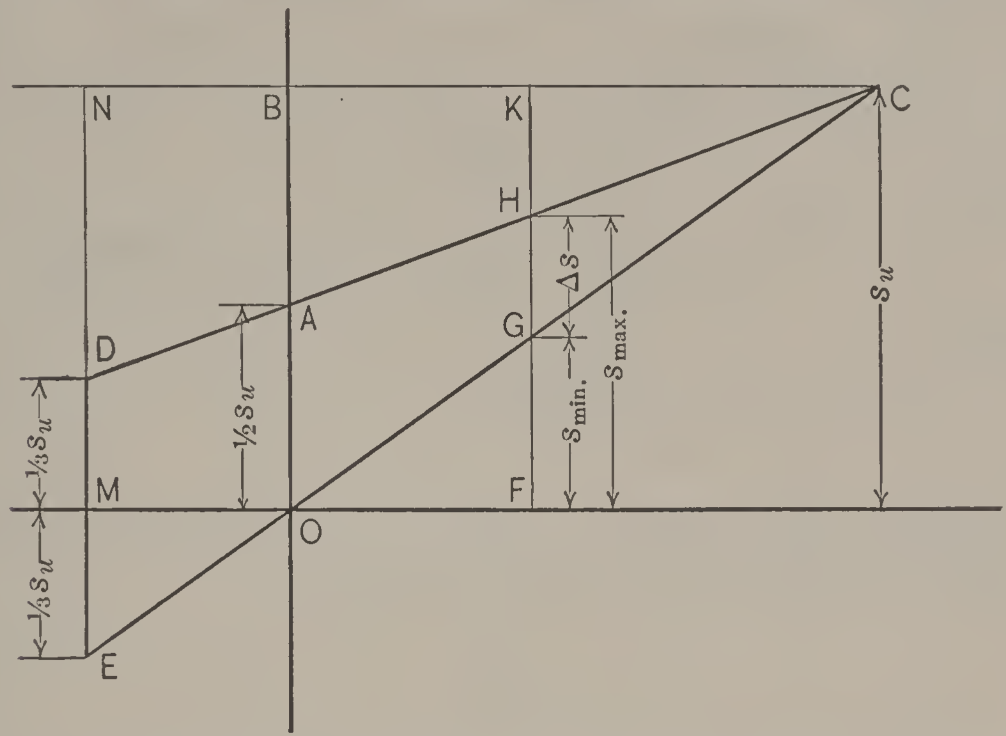

FIG. 247.—Goodman-Johnson diagram. Effect of range of stress on endurance limit of steel.

mum stress equal to zero the endurance limit was equal to about $\frac{1}{2}s_u$.

Now from Fig. 247 we have,

$$ND = DE, \qquad BA = OA, \qquad KH = HG = \Delta s, \text{ etc.}$$

Therefore,

$$\Delta s = FK - FH,$$

or

$$\Delta s = s_u - s_{max.} \qquad (237)$$

And since by definition, $\Delta s = s_{max.} - s_{min.}$, equation (237) may also be written:

$$\Delta s = \tfrac{1}{2}(s_u - s_{min.}). \qquad (238)$$

But $s_{max.}$ is the endurance limit corresponding to the range of stress Δs. If then s'_r is used to denote the endurance limit, equation (237) may be written

$$s'_r = s_u - \Delta s. \qquad (239)$$

[9] As already noted, the more recent tests show that this value is nearly $\frac{1}{2}s_u$, the higher ratio being due probably, in part at least, to more generous fillets and better surface finish.

It is frequently more useful to have the endurance limit expressed in terms of the ratio of the minimum unit-stress to the maximum unit-stress $\left(\frac{s_{min.}}{s_{max.}}\right)$ than in terms of the range of stress. Thus, equation (237) may be transformed as follows:

$$\begin{aligned} s_{max.} &= s_u - \Delta s \\ &= s_u - (s_{max.} - s_{min.}) \\ &= s_u - (s_{max.} - q s_{max.}), \quad \text{where} \quad \frac{s_{min.}}{s_{max.}} = q. \end{aligned}$$

Therefore,

$$s_{max.} = \frac{s_u}{2-q} = \frac{\frac{1}{2}s_u}{1-\frac{1}{2}q},$$

or since $s_{max.}$ is the endurance limit, s'_r, for the given ratio, q, of minimum to maximum unit-stress we have

$$s'_r = \frac{\frac{1}{2}s_u}{1-\frac{1}{2}q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (240)$$

which will be called the Goodman-Johnson formula.

The number of experiments made by Wöhler to determine the effect of range of stress was not large; recent experiments, as already noted, indicate that the Goodman-Johnson formula gives results on the side of safety.

Illinois Empirical Formula.—Since the endurance limits with completely reversed cycles of bending stress are now well established for a variety of steels, and can be found fairly easily from tests for any given steel (see Art. 156 for methods of testing) it is desirable to express the endurance limit of the steel for a given range of stress in terms of the endurance limit for completely reversed cycles of bending stress instead of the ultimate strength of the material.

From a series of tests made in connection with the investigation of the fatigue of metals at the University of Illinois, F. M. Howell found that the following empirical formula[10] expresses the relation between the endurance limit s'_r of the material when subjected to any given maximum and minimum stresses in each cycle

[10] Bulletin 124, Engineering Experiment Station, University of Illinois.

of bending stress and the endurance limit s_r of a material when subjected to completely reversed bending stress:

$$s'_r = \frac{q+3}{2} s_r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (241)$$

in which q is the algebraic ratio of the minimum stress to the maximum stress during a cycle of stress (for completely reversed stress $q = -1.0$). Equations (240) and (241), however, must be considered as tentative until more experimental data are available. (For a formula based on the energy theory see Bulletin 142, Engineering Experiment Station, Univ. of Illinois.)

It will be assumed, tentatively, that with cycles of shearing stress the effect of range of stress is the same as with cycles of bending stress. This assumption seems decidedly on the side of safety since the results of the tests [11] now available indicate that, with repeated shearing stress, range of stress has considerably less effect on the shearing endurance limit than does range of stress on the endurance limit with repeated bending stress.

Caution.—The limiting value of the unit-stress for any steel should never be taken greater than the yield-point even if the above formulas give a value of the endurance limit larger than the yield point. Thus, in the above equations the maximum value of s'_r to be used is the yield-point of the material. That is, if s'_r is found from equation (240) or (241) to be greater than the yield-point of the material the *static* strength of the member rather than the *fatigue* strength governs the design of the member. This fact should be clearly grasped since the value of s'_r, in the above equations, for machine members that are subjected to cycles of stress in which the stress does not change sign is seldom less than the yield-point of the material.

The Goodman-Johnson formula (or the Launhardt-Weyrauch formula which was similar to this) was formerly used extensively in designing bridge members. Static strength, however, will almost always govern the design of such members and the use of repeated stress formulas has practically been abandoned. It was frequently assumed that the use of a fatigue formula for bridge members was necessary to take account of the effect of impact loads. There was little justification, however, for using a fatigue

[11] Bulletin 142, Engineering Experiment Station, University of Illinois.

formula for this purpose and now, since allowance is usually made for impact loads and certain secondary stresses, the effect of fatigue is properly neglected.

ILLUSTRATIVE PROBLEM

Problem 242.—The pulley, A, (Fig. 248) exerts a constant twisting moment of 4800 lb. in. on the shaft B. The total bending load P due to the tensions in the belt on the two sides of the pulley is 800 lb. The diameter of the shaft is 2 in. and is mounted in flexible bearings. The shaft is made of cold-rolled steel for which the tensile ultimate strength is 80,000 lb. per sq. in. and the tensile yield-point is 40,000 lb. per sq. in. The value of the endurance limit of the material is assumed to be unknown. Is the shaft well designed?

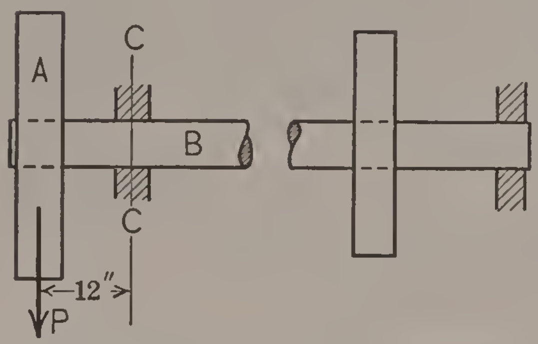

FIG. 248.

Solution.—Static Loading.—The stresses due to the loads, considering the loads to be static loads, are as follows: the shearing unit-stress due to the twisting moment is (see Art. 26),

$$s_s = \frac{Tc}{J} = \frac{4800 \times 1}{\frac{\pi(2)^4}{32}} = 3060 \text{ lb. per sq. in.}$$

The maximum fiber unit-stress at section CC due to the bending moment is (see Art. 34),

$$s = \frac{Mc}{I} = \frac{9600 \times 1}{\frac{\pi(2)^4}{64}} = 12{,}240 \text{ lb. per sq. in.}$$

The maximum combined normal unit-stress is (see Art. 109),

$$\text{max. } s' = \tfrac{1}{2}s + \tfrac{1}{2}\sqrt{s^2 + 4s_s^2}$$

$$= \tfrac{1}{2}12{,}240 + \tfrac{1}{2}\sqrt{12{,}240 + 4(3060)^2}$$

$$= 6120 + 6840$$

$$= 12{,}960 \text{ lb. per sq. in.}$$

and the minimum normal unit-stress is

$$\text{min. } s' = \tfrac{1}{2}s - \tfrac{1}{2}\sqrt{s^2 + 4s_s^2}$$

$$= 6120 - 6840$$

$$= -720 \text{ lb. per sq. in.}$$

As explained in Art. 109 *max.* s' and *min.* s' occur at the same point in the material, but on planes at right angles to each other. Further, when s is a tensile stress *max.* s' is a tensile stress and *min.* s' is a compressive stress, and when s is a compressive stress, *max.* s' is a compressive stress, and *min.* s' a tensile stress.

The maximum combined shearing unit-stress is (see Art. 109).

$$s'_s = \pm\tfrac{1}{2}\sqrt{s^2+4s_s^2}$$

$$= \pm 6840 \text{ lb. per sq. in.}$$

and this stress occurs at the same point where *max.* s' and *min.* s' occur but on each of two planes that bisect the angles between the planes on which *max.* s' and *min.* s' occur.

Since a tensile and compressive working unit-stress of at least 16,000 lb. per sq. in. and a shearing working unit-stress of 10,000 lb. per sq. in could be used with static loading the shaft has ample resistance to static loads.

Repeated Loads.—The number of repetitions of stress would be sufficient to cause rupture if the material is stressed above the endurance limit. The shaft has no abrupt change of section at the section, CC, of maximum stress but it will be assumed that the surface of the shaft is rather rough due to scratches caused by grit in the bearing, and it will be assumed that these scratches are in vertical planes and hence would cause localized bending stress only; the shearing stress, s_s, being unaffected.

Therefore, the value of s as found above must be increased. Thus, according to Table 4,

$$s = 1.2\times 12{,}240 = 14{,}700 \text{ lb. per sq. in.}$$

if this value of s is used in the above equations the following values for *max.* s', *min.* s' and s'_s are found

$$max.\ s' = 7350+7950$$

$$= 15{,}300 \text{ lb. per sq. in.,}$$

$$min.\ s' = 7350-7950$$

$$= -600 \text{ lb. per sq. in.}$$

$$s'_s = \pm 7950 \text{ lb. per sq. in.}$$

Now the failure of the shaft due to repeated stress may be due to (a) the completely reversed bending stress, (b) the completely reversed combined shearing stress, and (c) the combined normal stress that is not completely reversed.

(a) The stress s (14,700 lb. per sq. in.) is a *completely reversed* bending stress. But the endurance limit of the material with completely reversed bending stress may be taken as 0.45 of the tensile ultimate strength, and hence the above reversed stress is approximately 0.41 of the endurance limit, since

$$\frac{14{,}700}{0.45\times 80{,}000} = 0.41.$$

(*b*) The stress s'_s (7950 lb. per sq. in.) is a completely reversed shearing stress. Now the shearing endurance limit of the material is (see Art. 127).

$$(s_r)_s = s_r \times 0.55 = 36{,}000 \times 0.55 = 19{,}800 \text{ lb. per sq. in.}$$

Hence the reversed shearing stress is approximately 0.40 of the endurance limit since

$$\frac{7950}{19{,}800} = 0.40.$$

(*c*) The stress s' varies in each cycle from *min.* s' (-600 lb. per sq. in., compressive stress) to *max.* s' (15,300 lb. per sq. in., tensile stress). Now for this range of stress the endurance limit of the material is

$$s'_r = \frac{\frac{1}{2}s_u}{1-\frac{1}{2}q} \quad \text{or} \quad s'_r = \frac{3+q}{2}s_r. \qquad \text{(see Art. 132)}$$

If the first expression is used we have:

$$s'_r = \frac{\frac{1}{2}80{,}000}{1-\frac{1}{2}\left(\frac{-600}{15{,}300}\right)} = \frac{40{,}000}{1+\frac{1}{2}0.039} = \frac{40{,}000}{1.020} = 39{,}200 \text{ lb. per sq. in.}$$

And hence the maximum stress in each cycle is approximately 0.39 of the calculated value of the endurance limit, since

$$\frac{15{,}300}{39{,}200} = 0.39.$$

If the second expression is used we have:

$$s'_r = \frac{3+0.039}{2} \cdot 0.45 \cdot 80{,}000$$

$$= 54{,}700 \text{ lb. per sq. in.}$$

Hence the maximum stress in each cycle is approximately 0.28 of the calculated value of the endurance limit.

The critical stress, therefore, is the completely reversed bending stress, and its value is somewhat too high unless the uncertainty in the loads is relatively small.

PROBLEMS

243. A chrome-nickel heat-treated crank shaft of a single acting gas engine is subjected to repeated applications of a torsional shearing stress which varies approximately from zero to a maximum ($q=0$); bending stresses may be neglected. If the shaft is 2 in. in diameter, what maximum torque should be allowed? Use values in Table 7.

244. If in Problem 242 the distance from the action line of P to the section CC were 8 in. and the twisting moment were 6000 lb.-in., what stress would approach closest to the endurance limit?

245. A special chrome-vanadium heat-treated steel having an ultimate tensile strength of 200,000 lb. per sq. in. and proportional limit of 150,000 lb. per sq. in. is used in a helical spring on an automobile truck. The stress in the spring is mainly shearing stress (see Art. 90) and varies from a small value due to the weight of the truck to a maximum, ($q=0$). The maximum stress is not constant, however, since the impact loads vary greatly. It may be assumed that the spring is subjected frequently to overload stresses. Estimate the endurance limit and find a working stress, assuming that the working stress should not be greater than one-fourth of the endurance limit.

Ans. $(s'_r)_s = 74{,}400$ lb. per sq. in.; $s_w = 18{,}600$ lb. per sq. in.

PART II. MECHANICAL PROPERTIES OF STRUCTURAL MATERIALS

133. Introduction.—The effective use of materials in engineering structures and machines for resisting loads requires a knowledge of (1) the loads to which the members of the structure or machine are to be subjected, (2) the relations between the loads and the stresses and deformations caused by the loads; these relations involve the dimensions and form of the body or member and are discussed in Part I, and (3) the mechanical or physical properties of structural materials.

A knowledge of the properties of materials is needed for two purposes: (*a*) for use in establishing the relations under (2) and in the application of those relations to problems of design, and (*b*) for use in selecting materials best suited to the service requirements.

A discussion of the relations under (2) and a very brief discussion of the topics under (3) have been given in Part I. A more extended discussion under 3(*b*), that is, of the properties needed in materials for load-resisting members, as used in various types of structures and machines, and of methods of measuring the properties, is given in the following pages.

CHAPTER XV

DEFINITIONS, METHODS OF MEASURING, AND SIGNIFICANCE OF MECHANICAL PROPERTIES

134. Properties to be Considered.—A knowledge of the following properties is of special importance to the engineer in the use of materials in load-resisting structures and machines, and they can be measured quantitatively with a fair degree of satisfaction by means of mechanical tests.

1. Strength
   (*a*) Static
   (*b*) Impact and Energy
   (*c*) Fatigue
2. Ductility.
3. Stiffness
4. Resilience
5. Toughness
6. Hardness

On the other hand the properties, plasticity, malleability, and machineability are of importance mainly to the manufacturer in forming (rolling, forging, drawing, pressing, machining, etc.) the material for use in structures and machines, and their measurement is less definite than that of the properties mentioned above. However, these properties are closely connected with some of those mentioned above, as are also the general properties, flexibility and elasticity.

135. Meaning of Strength.—By strength of a material is meant that property which enables the material to resist external forces or loads without incurring structural damage. By *structural* damage is meant stress and deformation in the material of a member that cause the member to cease to function properly in the structure or machine; it has different meanings under different conditions of use or service; for example, a stress or a deformation that would cause structural damage in the member of a building might not cause structural damage in a chain hoist or a coal-car frame, and, on the other hand, a deformation that would cause structural damage in a lathe, planer, or other machine tool would not, in general, cause damage in a building, etc. Thus, the *maximum usable strength* of a material depends on the service in which the material is used, and accordingly it will be found desirable to have at least two measures of the maximum usable strength, namely, the *ultimate strength* and the *elastic strength.* However, no single quantity can adequately measure either the ultimate strength or the elastic strength of a material, for:

(1) The strength of a material depends on the type of loading (static, impact or energy, and repeated) to be resisted. Thus, a cast-iron beam is stronger in resisting static loads than is a similar oak beam, whereas the oak beam is stronger than the cast-iron beam when the load applied is an impact or energy load (see Chap. XIII). Again, a bar of steel may resist a static load of a given magnitude without incurring *measurable* structural damage and yet it may rupture when subjected to a load of the same magnitude when applied a large number of times (see Chap. XIV).

2. For any type of loading (static, impact and repeated) the strength of a material is different for different kinds of stresses (tensile, compressive and shearing) developed in the material. Thus the strength of cast iron, concrete, brick, and other brittle materials in tension and in shear are much less than their strengths in compression, and for some brittle materials including cast iron the shearing strength is greater than the tensile strength. Further, most ductile materials such as wrought iron, steel, soft brass, etc., are much weaker in shear than in tension and in compression.

The strength of a material may, therefore, be considered in accordance with the following outline:

Strength:

1. Static;	Elastic Ultimate	Tensile Compressive Shearing
2. Impact and Energy;	Elastic Ultimate	Tensile Compressive Shearing
3. Fatigue;	Ultimate	Tensile Compressive Shearing

Energy and fatigue strengths have already been discussed briefly in Chapters XIII and XIV, respectively.

Static Strength

136. Static Ultimate Strength.—The resistance that a material offers to a static (gradually applied) load, that is, the static strength of the material is measured in terms of the unit-stress (internal force per unit area) developed in the material. The static *ultimate* strength of a material is the unit-stress developed in the material by the maximum static load that the material can resist without rupturing.

Method of Determining Ultimate Strength.—The static ultimate strength of a material is determined by testing a specimen of the material (or several specimens) in a testing machine (see Fig. 3 for a common type of machine) which weighs the load applied to the specimen; the largest load applied before fracture occurs divided by the original area is taken as the measure of the ultimate strength. In determining the ultimate strength from a test, whether it be the tensile, compressive, or shearing ultimate strength, care must be exercised, particularly with brittle materials, in securing an axial load (Art. 3), that is, a load which causes a uniformly distributed stress on the cross-sectional area of the test specimen, since the value of s, as found from the equation

$$s=\frac{P}{a},$$

is the average unit-stress and not the maximum unit-stress developed in the specimen if the maximum load P does not cause the stress to be distributed uniformly on the cross-sectional area.

Experience has shown that the ordinary spherical-seated grips (see Fig. 249a) used on testing machines for tension and compression tests do not insure that the load shall be axial; for *ductile*[1] materials, however, the eccentricity introduced by such grips will not seriously affect the value of the ultimate strength. But, for *brittle* materials such as cast iron, cast aluminum alloys, etc., the eccentriticy may lower the value of the ultimate strength appreciably. In testing cast aluminum alloys it has been found[2]

[1] The property of ductility is discussed in Art. 141, but for use in this article it will be sufficient to define ductility as that property which enables the materials to draw out or become plastic under load; brittleness is the lack of ductility.

[2] Report on the Materials of Construction used in Aircraft and Aircraft Engines, Aircraft Production Department of the Ministry of Munitions, British Government.

that unless special loading devices (see Fig. 249*e*) are used the ultimate strength obtained is frequently 30 per cent too low and that the low values are obtained consistently if a definite routine is followed in putting the specimens in the grips.

Other tests[3] have shown that, even with approved methods of holding the specimen, the maximum unit-stress along an element or fiber of the specimen may be 10 per cent or more in excess of the mean unit-stress over the cross-sections.

Form of Test Specimen.—The form of the test specimen also has an influence on the distribution of stress on the cross-section within the gage length. For relatively ductile materials the form of grips and of specimens shown in Fig. 249(*c*) and (*d*) give satisfactory results. In a long tension specimen the unevenness of stress caused by the gripping device has a better opportunity to become nearly uniform on the cross-section within the gage length than in a short specimen. Therefore, spherical seated bearing blocks (Fig. 249*a* and *b*) are used, as a rule, only on short specimens. Short tension specimens are either threaded (Fig. 249*a*) or have shoulders on the ends (Fig. 249*b*); the latter method is used particularly with specimens that are to be tested in a hardened condition after being heat treated. The loading device shown in Fig. 249(*e*) is used in testing very brittle material. (For compression and shearing test specimens see Art. 139 and 140.) The significance of the ultimate strength of a material to the engineer is discussed in Art. 138. Values of ultimate strengths for various materials are given in the tables of Chapter XVI.

137. Static Elastic Strength.—In many structures and machines structural damage to the material occurs at stresses much below the ultimate strength of the material, that is, the maximum *usable* strength of the material is not, as a rule, best measured by the ultimate strength. There is need, therefore, for a measure of the unit-stress at which structural damage begins or at which only a small amount of structural damage has occurred.

The static elastic strength of a material is the maximum unit-

[3] "On the Stress Distribution During Tension Test," Engineering (Lon.) Dec. 10, 1907, p. 796; Oct. 29, 1909, p. 593. For tension tests of specimens held in spherical-seated holders the ratio of maximum stress to mean stress averaged 1.165; for compressive tests of specimens with spherical seated bearing blocks the ratio averaged 1.059; for both series of tests the average was 1.101.

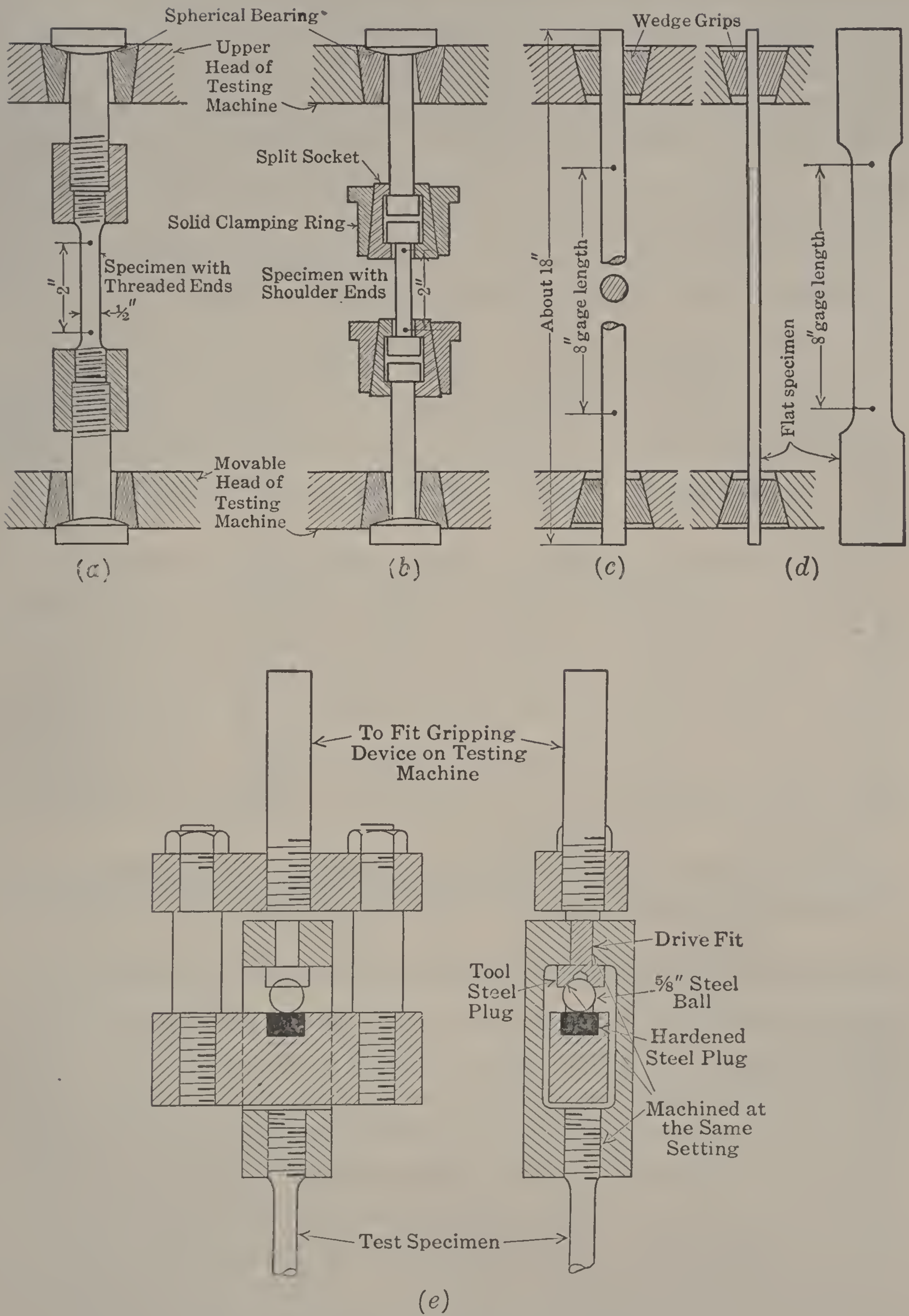

FIG. 249.—Methods of testing material in tension.

stress that can be developed in the material without causing appreciable structural damage. The criterion for structural damage is somewhat indefinite, but, in general, structural damage, *when caused by static loading*,[4] is closely associated with plastic action in the material, and the unit-stress at which plastic action is considered to begin or to have reached a measureable amount is usually taken as the measure of the static elastic strength of the material. In an actual test of a material (whether in tension, in compression or in shear) the determination of the unit-stress at which plastic action begins is subject to uncertainties, and hence a number of limiting stresses have been proposed and used as measures of the static elastic strength of a material; the various limiting stresses differing mainly by the amount of plastic action that is considered to be most significant or most convenient to measure.

The following measures of the static elastic strength of a material are discussed below: proportional limit, yield point, elastic limit, Johnson's apparent elastic limit, A.S.T.M. elastic limit, and proof stress. Further, although it is assumed, for convenience, that the material is tested in tension, the discussion will also apply to a compression and a shearing test. Values of the static elastic strengths of various materials are given in the tables of Chapter XV.

(*a*) *Proportional Limit.*—If a bar of mild steel is subjected to a gradually increasing axial load P (Fig. 250*a*), the unit-stress s, equal to $\frac{P}{a}$, and the unit-elongation (unit-strain) ϵ, equal to $\frac{e}{l}$, will increase so that the unit-stress is proportional (within closely approximate limits) to the unit-strain (Fig. 250*b*), until a unit-stress called the proportional limit (sometimes called proportional elastic limit) is developed in the material; the proportional limit is represented by $A'A$ in Fig. 250(*b*) and (*c*). The curve of Fig. 250(*b*) is called a stress-strain diagram and is obtained by plotting the unit-stresses and corresponding unit-strains as found from the test data. The unit-strains that occur before the proportional limit is reached, represented by the abscissa to the first part of the stress-strain curve in Fig. 250(*b*), are plotted to a larger scale

[4] Material may rupture when subjected to repeated loads without giving any evidence of plastic action, as is discussed in Chapter XIV and also in Art. 155.

in Fig. 250(*c*) in order better to determine the modulus of elasticity (or the slope of the curve OA) and the proportional limit (or the point of tangency of OA and ABC), and also to bring out more clearly the form of the part ABC of the curve, since the strains corresponding to this part of the curve are closely connected with the beginning of structural damage.

The *proportional limit* of a material, then, is the greatest unit-stress to which the material may be subjected without causing the unit-strain to increase at a faster rate than does the unit-stress.

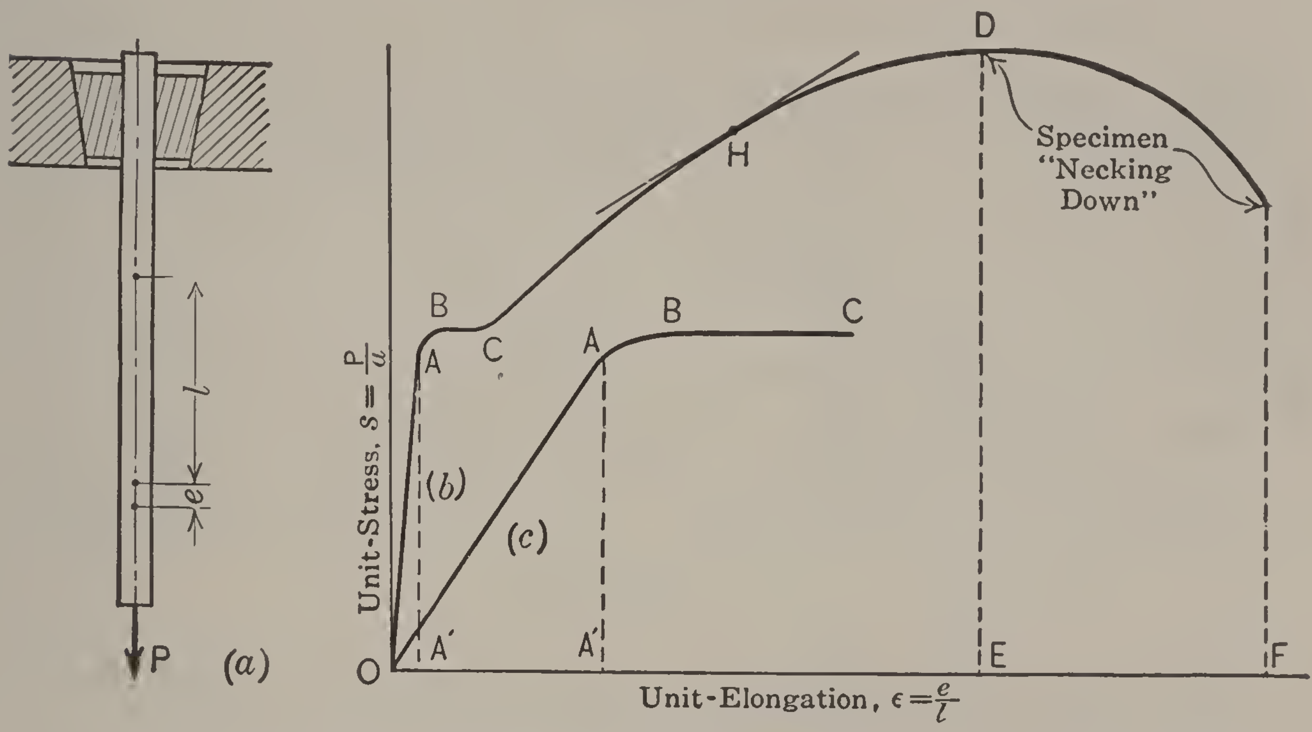

FIG. 250.—Stress-strain diagram of ductile steel; (*b*) the complete diagram, (*c*) the first part of diagram using a large scale for unit-elongations.

The questions that now arise may be stated as follows: Is the proportional limit of a material, as determined from test data, the unit-stress at which plastic action (structural damage) *begins*? Is it a reliable measure of the maximum usable stress for material as used in structures and machines? In order to answer these questions the conditions or factors in the test which influence the value of the proportional limit must be investigated; and further, the extent to which these conditions influence the usefulness of the material in the structure or machine must be considered.

Several of these influencing factors are discussed as follows:

1. The proportional limit may be lowered appreciably by unevenness of stress in the fibers of the test specimen:

(*a*) Unevenness of stress may be due to eccentricity of loading caused by the gripping devices, etc. For example, in Fig. 251 are shown the tension stress-strain curves obtained by plotting the strains measured on three independent gage lines (120° apart) against the corresponding average unit-stresses on the cross-section of the specimen, when the out-of-alignment of the jaws of the machine (the eccentricity of loading) was 0.035 in. These curves indicate the great unevenness of deformation (and hence of stress) due to eccentricity of loading. Such unevenness of stress tends to lower the proportional limit and to produce a long bend for the portion AB (Fig. 250c) in the stress-strain curve indicating

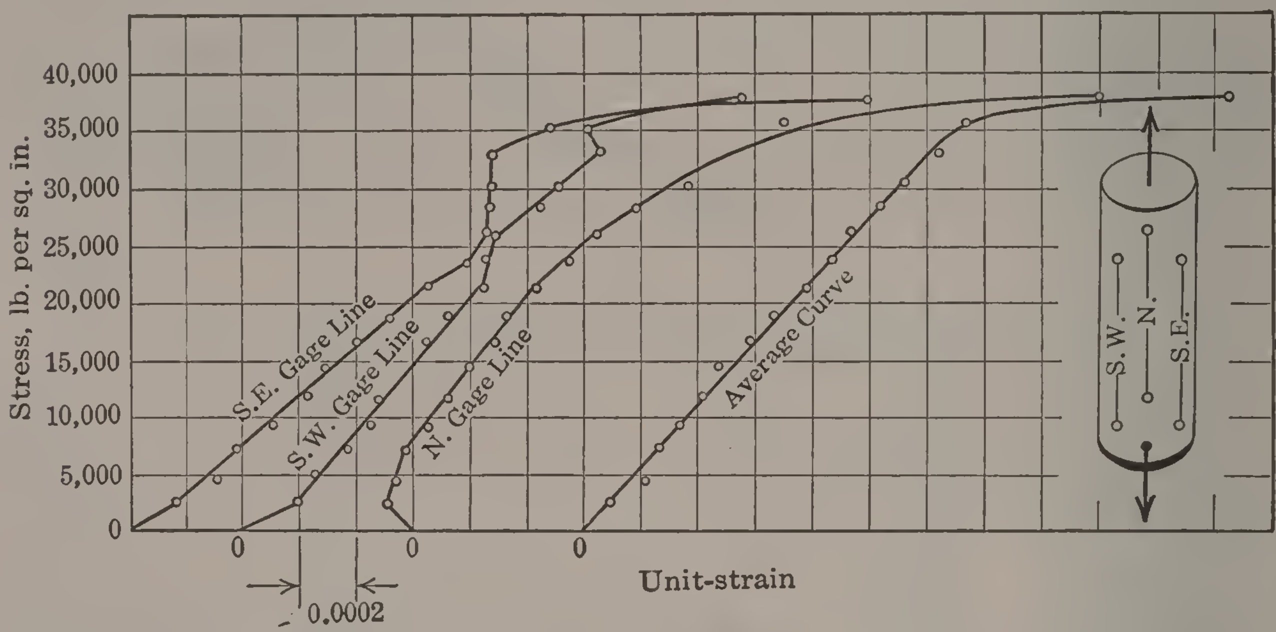

Fig. 251.—Stress-strain diagrams showing unevenness of stress in specimen held in wedge grips when one pair of grips was 0.035 in. out-of-line.

plastic action over a wide range of stress. The curve found by plotting the average of the strain readings on the three-gage lines is also shown in Fig. 251; this curve gives but little evidence of the uneven stretching of the specimen, and hence with the usual extensometer, which averages the deformations on two or more gage lines, eccentricity of loading will not necessarily be detected by noticeable irregularities in the stress-strain diagram.

(b) Unevenness of stress may be due to initial internal stresses caused by hardening steel, bronze, etc., by heat treatment or by cold working as in cold drawing and rolling. The stress-strain diagram for steel containing initial stresses is usually of the form shown in Fig. 252(a), and after the stresses are released by heating the steel mildly and cooling relatively slowly (tempering) the curve takes the form shown in Fig. 252(b); the increase in the proportional limit being large; in some cases much the same effect can be produced by subjecting the steel to several thousand repetitions of low stress.

2. The more sensitive the extensometer is, the lower on the curve will the proportional limit or point of tangency, *A* (Fig. 250*c*), be detected, other conditions being the same. Thus, in one series of tests [5] the results given in Table 10 were obtained, three specimens being taken from each of three different bars of soft steel of supposedly the same quality.

TABLE 10

Smallest Unit-deformation Indicated by Extensometer	Specimen from Bar No.	Proportional Limit. Lb. per sq. in.	Average Value for the Three Specimens in Each Group
0.000,000,27...........	1	38,260	
	2	32,670	32,700
	3	27,160	
0.000,000,5...........	1	40,460	
	2	35,250	34,480
	3	27,740	
0.000,012,5...........	1	41,320	
	2	34,590	38,180
	3	38,620	

Thus, the average of the proportional limits of the specimens on which the most sensitive extensometer was used is 5480 lb. per sq. in. lower than that of the specimens on which the least sensitive (but commonly used) extensometer was employed.

3. The value of the proportional limit of metals is also affected by the crystalline structure of the material. For example, metal having large or coarse grains due to slow cooling from a high temperature (as in the case of steel and brass castings, etc., or of rolled and forged steel on which the rolling or forging has been stopped at a temperature considerably above the critical temperature) has a relatively low proportional limit since the stress-strain diagram is curved over a considerable distance along the curve as

[5] "The Physical Significance of the Elastic Limit," by H. F. Moore. Proc. Inter. Assoc. for Testing Materials, Vol. XXVIII.

indicated in Fig. 252(a), and any treatment which will refine the grain (such as heat treatment, or mechanical working of the metal while it is cooling to the critical temperature) may materially increase the proportional limit, as is shown in Fig. 253.

4. The stress-strain diagrams for some brittle materials such as cast iron, concrete, etc., are curved practically from the start, similar to that shown in Fig. 252(a) (see Fig. 261), and hence the proportional limit for these materials is practically zero.

5. When the stress-strain curve bends away from the straight line very gradually as in Fig. 252(a) and 253(a) the determination

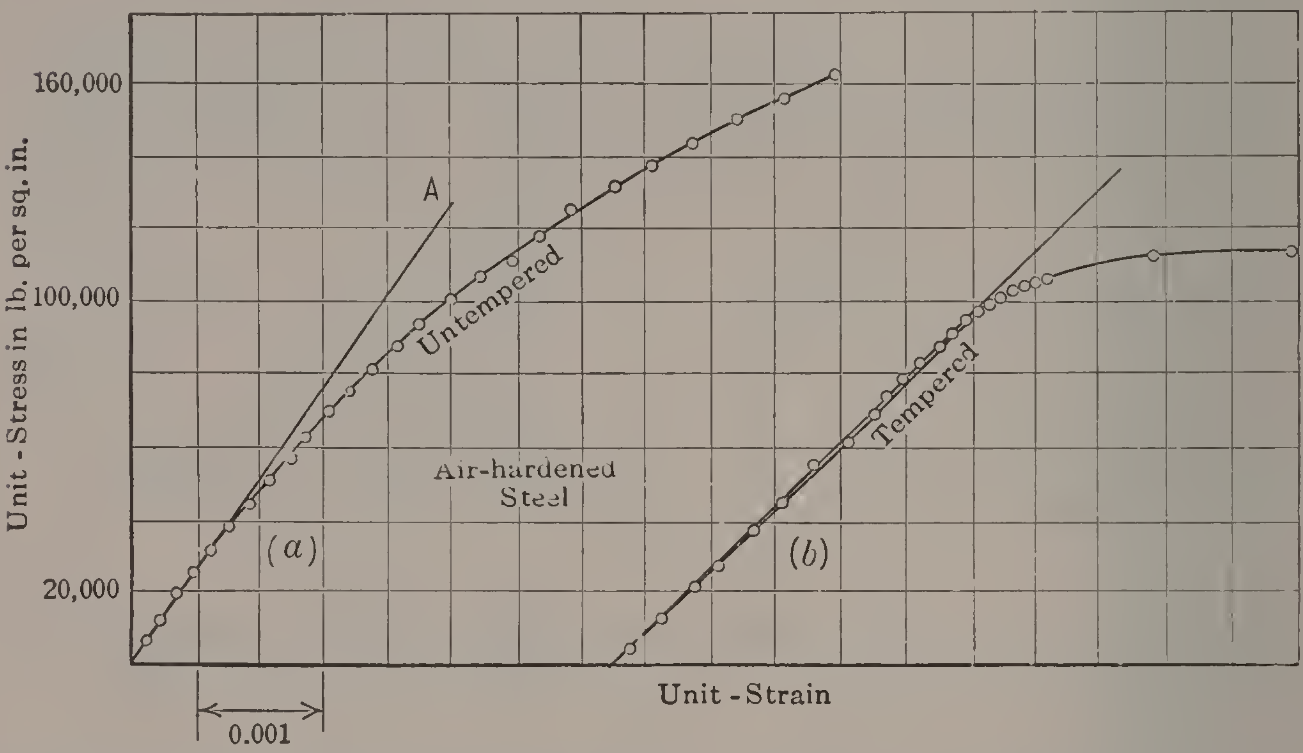

FIG. 252.—Effect of internal stress due to hardening by heat treatment on proportional limit of steel.

of the point of tangency of the curve and the straight line (proportional limit) is subject to considerable variation and is also dependent upon the scales used in plotting the stresses and strains.

Summary.—The value of the proportional limit of a material as found from a test is influenced (a) by local yielding of the specimen (localized strains) due to a slight eccentricity of load, initial stresses, etc., particularly when determined with an extensometer that will detect very small deformations: (b) by the crystalline or internal structure of the material due to methods of cooling and of mechanical working, the value of the proportional limit for some coarse-grained brittle materials being practically zero.

Is the Proportional Limit a Reliable Measure of the Maximum Usable Stress? The question now arises as to whether it is desirable to have a measure of the maximum usable stress that is influenced by the factors discussed above. It is undoubtedly true that the material in a member of a structure or machine is seldom if ever free from localized stresses and strains sufficient to cause local yieldings when the member is subjected to its working or design loads, these local yieldings, however, do not as a rule cause structural damage to the member *as a whole* provided that the material has sufficient ductility to permit the local yielding to occur and

Fig. 253.—Stress-strain diagrams of steel casting and rolled steel showing the effect of the hot-working. (*a*) Specimen from steel casting that cooled slowly after being poured. (*b*) Specimen of same chemical composition as (*a*), steel rolled while cooling.

thereby relieve the material of the high localized stresses. Hence, the maximum usable strength of a ductile material is likely to be greater than the proportional limit. Further, brittle material such as cast iron and concrete can be used safely when subjected to stresses above their proportional limits.

It appears, therefore, that the proportional limit may be regarded as a reliable measure of the maximum usable strength of a *ductile* material provided that it is found from a specimen having a minimum of initial stress and that the specimen is tested with a minimum of eccentricity of loading so that the first yielding is also a rather general yielding of the whole specimen, as indicated by an abrupt type of stress-strain curve.

Much of the hot-rolled and forged structural steel (and other hot worked ductile metals) which has been finished close to the critical temperature is fairly free from internal stresses and is fine grained; and, if tested with standard gripping devices (see Fig. 249) and strain-measuring apparatus, the proportional limit is a fairly satisfactory measure of the beginning of real structural damage of such material. However, if the rolling has been stopped at too high a temperature as sometimes occurs with heavy rolled sections, plates, etc., the crystalline structure will be coarse and the stress-strain curve will be of the gradual type as shown in Fig. 252(a) and 253(a) instead of the abrupt type shown in Fig. 252(b) and 253(b). Or, if the rolling has been finished at too low a temperature as is likely to occur with the thinner rolled-sections and with plates, initial stresses may be set up in the material, due to the cold working, which may cause the proportional limit to be less than the stress that would cause structural damage in the structure or machine in which the material is used. Further, for most hard steels and other brittle materials in general, the proportional limit is not a satisfactory measure of the static elastic strength. And, in any case, the uncertainties as to the real value of the proportional limit make desirable the measurement of the elastic strength by means of a unit-stress at which there occurs a small but measurable plastic deformation which is an indication of *incipient general yielding* of the specimen rather than by the proportional limit which is likely to be an indication of the *beginning of local yielding* that may not be indicative of structural damage under service conditions. Thus, the yield-point, Johnson's Apparent Elastic Limit, etc., will now be considered as measures of the maximum usable strength of a material.

(b) *Yield-point.*—The yield-point of a material is the unit-stress at which the material yields appreciably without an increase of load. In Fig. 250 the yield-point is represented by the ordinate to the horizontal line BC. Only ductile materials have yield-points. It is evident that structural damage to most structural and machine members has occurred when the primary stresses in the member reach the yield-point of the material, since the member will be permanently distorted.

At stresses below the proportional limit the deformation is practically all elastic, whereas at the yield-point the deformation is mainly plastic yielding. At stresses between the proportional limit and the yield-point (between points A and B, Fig. 250c) the deformation is partly elastic and partly plastic. If the change from elastic to plastic deformation is very gradual the ratio of the yield point to the proportional limit is large. This ratio is sometimes used as a measure of the amount of internal stress in the material and hence as a measure of the degree of success in carrying out various heat treatments of steel, since the proportional limit of a ductile steel free from internal stress and tested under an axial load is practically the same as the yield-point.

Methods of Determining the Yield-point.—The yield-point is usually determined in commercial testing by reading the load indicated by the poise on the beam of the testing machine (see Fig. 3) when the beam drops and remains down for some time as the specimen continues to stretch, thus indicating appreciable stretch without increase of load. It may also be found by use of dividers; the dividers are set to span a relatively short gage length on the specimen and are held on the specimen, and the least load that causes the gage length to become greater than the span of the dividers is noted and taken to be the yield-point load.

The value of the yield-point is affected less than is the proportional limit by the conditions discussed under (a) above, but when determined by the " drop of the beam " the value may be raised from 3000 to 5000 lb. per sq. in., by the manipulation of the operator, by moving out the poise rapidly just as the yield-point is approached and thus taking advantage of the inertia of the beam. For a discussion of the determination of the so-called yield-point for non-ductile materials, see Art. 137(f).

Is the Yield-point a Reliable Measure of the Maximum Usable Stress? The question now arises as to whether the yield-point, if carefully determined, is a reliable measure or criterion of the static elastic strength of a material to be used in structures and machines. The material in some structures and machines is not damaged if the stress developed is greater than the yield-point but for most structural uses the primary stresses in the members must be kept below the yield-point if the members are to perform their function in the structure. In such service and for ductile material the yield-point may be regarded, as the maximum usable strength of the material.

The yield-point, therefore, is a reliable measure of the minimum stress at which real structural damage becomes evident and hence of the maximum usable stress provided: (1) that the material is ductile enough to possess a real yield-point, (2) that it is carefully determined, and (3) that the material is in such a condition (free from internal stress, etc.) that the change from elastic to plastic deformation is rather abrupt (Fig. 253*b*) instead of gradual (Fig. 253*a*).

If a material is ductile, is free from internal stresses (such as most rolled and forged low carbon steel), and is loaded axially when tested, the proportional limit and the yield-point are practically the same value, and either one is a satisfactory measure of the static elastic strength; whereas, for metals that have been so treated as to produce a coarse-grained structure or to produce internal stresses (as in hardening by heat treatment or in cold working) there may be a wide difference between the values of the proportional limit and yield-point, and under such conditions neither the proportional limit nor the yield-point is as reliable a measure of the maximum usable strength of the material as is frequently desired. Further, neither of them is a satisfactory measure of the elastic strength of most brittle materials. Hence, there is need of a method of determining a stress between the proportional limit and the yield-point at which the deformation is partly elastic and partly plastic, with sufficient plastic action to be readily measurable but sufficient only to indicate *incipient* general yielding of the material. Several such measures of the elastic strength are discussed under (*d*), (*e*), and (*f*) below; before discussing these, however, the elastic limit of a material will be considered.

(*c*) *Elastic Limit.*—Permanent deformation or set is fundamentally the best evidence of plastic action or of the breakdown of elastic action in the material. The maximum unit-stress that can be developed in the material without causing a permanent set is called the *elastic limit* of the material. It may be found by applying a small load to the specimen and then releasing the load and noting whether the extensometer reading returns to zero; if there is no permanent set a larger load can be applied and so on, until a slight permanent set is indicated by the extensometer. If still larger loads are applied and then released, the sets will be found to increase similar to that indicated by the *stress-set* curve in Fig. 254(*a*).

The value of the elastic limit as found from tests is affected by the same conditions (eccentricity of load, sensitiveness of extensometer, initial stresses, etc.) as is the value of the proportional limit, and permanent sets may frequently be detected at low stresses particularly with sensitive extensometers but such sets, due largely to initial and localized stresses, although present in the members of many structures and machines, do not seriously affect the strength of the member as a whole, provided that the material is relatively ductile. And hence such preliminary sets

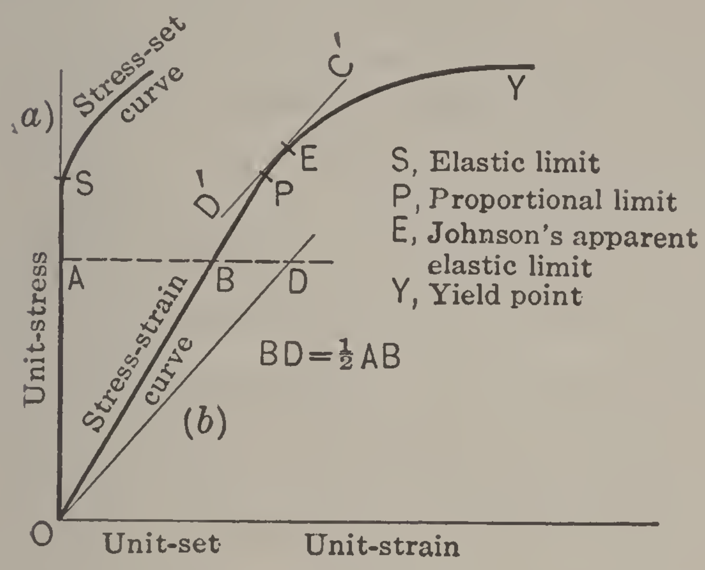

FIG. 254.—Methods of determining the maximum usable strength of a ductile material.

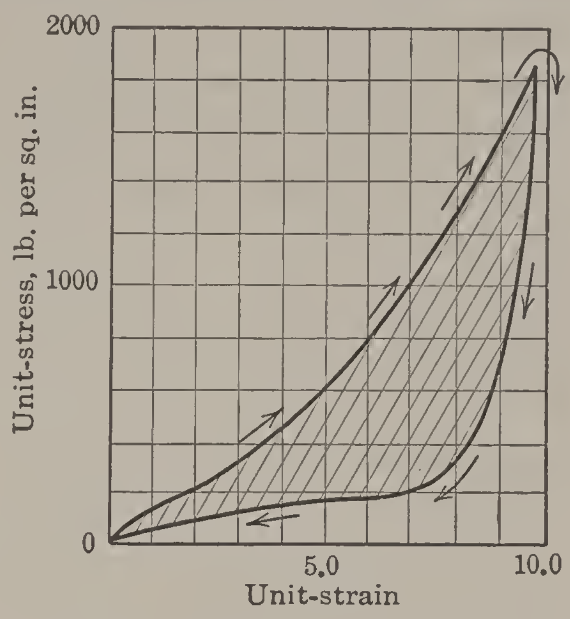

FIG. 255. — Tensile stress-strain diagram for rubber; maximum unit-stress in cycle about 0.70 of ultimate strength.

are, as a rule, not a reliable measure of structural damage when the member is subjected to static loads.

For ductile metals the value of the proportional limit and elastic limit *as found from tests* are practically equal. There is, however, no fundamental relation between the proportional limit and the elastic limit of a material. In other words, there is nothing in the property of elasticity which requires that unit-stress shall be proportional to unit-strain; a material that has a curved stress-strain diagram might retrace that curve on release of the stress and hence have perfect elasticity. In fact, the stress-strain curve for rubber is a reversed curve (Fig. 255) and on release of the load (if the load is not too large) the material shows no permanent deformation although the curve obtained during release of load is not coincident with the curve obtained during application of load, that is, there is a hysteresis loop.

However, although there is no fundamental relation between the proportional limit and the elastic limit, tests indicate that for all wrought structural metals and for practically all structural materials a deviation from a straight stress-strain diagram is accompanied by plastic deformation or set. For most structural materials, therefore, the proportional limit serves the same purpose as does the elastic limit and is much easier to determine.

(*d*) *Johnson's Apparent Elastic Limit.*—In order to avoid the objections to both the proportional limit and the yield-point, as discussed under (*a*) and (*b*) above, J. B. Johnson [6] used a unit-stress called the *apparent elastic limit* which is the unit-stress at which the deformation increases, with respect to stress, at a rate 50 per cent greater than that at zero (or small) stress. It is found by drawing a line OD (Fig. 254) on the stress-strain diagram which has a slope 50 per cent greater than that of the straight part of the curve below the proportional limit, and then drawing a line $D'C'$ parallel to OD and tangent to the stress-strain curve; the ordinate to the point of tangency is the unit-stress at which the deformation is increasing, with respect to stress, 50 per cent greater than at any stress below the proportional limit.

The value of 50 per cent increase in slope is arbitrarily selected, the purpose being merely to select, by a convenient method, a unit-stress at which a small but measurable deviation from the straight stress-strain line which is indicative of the beginning of a general plastic yielding of the specimen.

A unit-stress selected by the same method but by using a value of 100 per cent increase in slope, instead of 50 per cent increase has been called the *useful limit point*.

The method of determining Johnson's apparent elastic limit is applicable to nearly all forms of stress-strain curves obtained from structural materials and the value obtained is probably the best *single* measure of the maximum usable stress (static elastic strength) of a structural material.

(*e*) *A.S.T.M. Elastic Limit.*—As stated under (*b*) above, the yield-point is usually a satisfactory measure of the elastic strength of a ductile material, provided that reasonable care is taken in securing an axial load in testing, and that the material is reasonably free from internal stresses, etc., so that the yield-point is sharply defined. Many materials, such as heat-treated or cold-rolled

[6] Johnson's Materials of Construction, 5th Edition.

medium-carbon and alloy steels used for axles, etc., however, do not have a sharply defined yield-point, and for such materials a measure of static elastic strength is found, particularly in commercial tests which determine acceptance or rejection of the material, by the method prescribed in some of the specifications of the American Society for Testing Materials, described as follows: "The elastic limit called for by these specifications (for heat treated axles and shafts, cold-rolled axles, etc.) shall be determined by an extensometer reading to 0.0002 in. The extensometer shall be attached to the specimen at the gage marks and not to the shoulders of the specimen nor at any part of the testing machine. When the specimen is in place and the extensometer attached the testing machine shall be operated so as to increase the load on the specimen at a uniform rate. The observer shall watch the elongation of the specimen as shown by the extensometer and shall note, for this determination, the load at which the rate of elongation shows a sudden increase."

The stress found by this method, as a rule, will be represented by a point on the stress-strain curve somewhat above Johnson's apparent elastic limit and hence should be considered perhaps as an accurate and conveniently made determination of the yield-point; it has decided advantages over the method of determining the yield-point by the "drop of the beam" or by dividers particularly for the class of material that exhibits a gradual increase in plastic yielding. However, for high-carbon steel, cast iron, and other metals of a like degree of brittleness, the "sudden increase" referred to above is not well marked, and for such material this determination tends to yield unsatisfactory results.

(*f*) *Proof Stress.*—As has already been stated the yield-point is defined to be the unit-stress at which deformation occurs without increase of load and that one of the methods of determining the yield point is to obtain the load (and hence the unit-stress) at which a distinctly visible increase occurs in the distance between gage points on the test specimen as observed by using dividers. Now such a visible increase in deformation may be observed in material which, according to the definition has no yield-point, that is, the deformation observed is elastic deformation; this is particularly true in testing high-strength steel, when using a relatively long gage length (4 in. or over). Such a use of the yield-point is considered objectionable since the maximum usable stress may be greater than the value found in this way. Hence a stress called the proof stress has been proposed and adopted, in British specifications, for certain classes of material as a substitute for the yield-point. The *proof stress* is a unit-stress applied for a short time (fifteen seconds, say) which must not produce a permanent set of more than a specified amount ($\frac{1}{2}$ per cent, say) when the stress is removed; that is, a permanent set less than the specified amount is not considered to be evidence of structural damage. Thus, in Fig. 256 the proof stress for the material is represented by the ordinate to the point P on the stress-strain diagram.

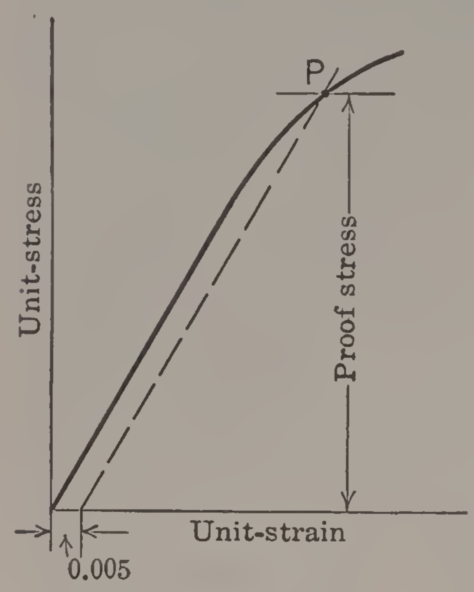

Fig. 256.

QUESTIONS

1. Does any single quantity measure adequately either the elastic strength or the ultimate strength of a material? If not, why?

2. What is meant by "structural damage"?

3. What do static elastic strength and maximum usable stress mean?

4. Name several measures of the static elastic strength of a material.

5. State and discuss several conditions which affect the value of the proportional limit as found from a test.

6. Should a measure of static elastic strength be influenced by the above conditions?

7. State several causes of unevenness of stress in a tension test specimen.

8. How does unevenness of stress affect the form of the stress-strain diagram?

9. Under what conditions is the proportional limit a reliable measure of the static elastic strength of a material?

10. Define yield-point of a material.

11. Of what use is the ratio of the proportional limit to the yield-point?

12. What advantages has the yield-point as a measure of static elastic strength over the proportional limit? Under what conditions are the values of the proportional limit and yield-point approximately equal?

13. What kinds of material have no yield-point?

14. Define elastic limit. Is there any fundamental relation between the elastic limit and proportional limit of a material? Is there any relation as found from experiment? If so, what relation?

15. Define Johnson's apparent elastic limit.

16. What are the advantages of Johnson's elastic limit as a measure of elastic static strength of a material?

17. Define the A.S.T.M. elastic limit. For what class of materials is it specified? Why?

18. What is the proof stress? Why has it been used in preference to the yield-point?

138. The Significance of the Ultimate and Elastic Strengths.— The material in some structural and machine members are frequently subjected to stresses greater than those proposed above as measures of the elastic strength of the material as, for example, freight-car frames, hoisting cables, ball bearings, etc.; the plastic flow of the material which accompanies these stresses do not necessarily destroy the usefulness of these members although the " life " of the members may be reduced because of such use. On the other hand, the distortion of the members in many (most) structures when stressed, as a whole, above the elastic strength of the material will destroy the usefulness of the structure or machine; this is true of permanent buildings, bridges, railroad rails, machine frames, etc.

If, then, members *in their normal service* are to be subjected to stresses above the elastic strength of the material the ultimate strength may be the maximum usable strength of the material. Further, in the case of brittle materials, which have no well-defined elastic strength, the ultimate strength is usually a fairly satisfactory measure of the maximum usable strength of the material regardless of the use of the material in the structure or machine, since for such material the deformation accompanying any stress below the ultimate strength is mainly elastic and hence the ultimate strength is also the elastic strength.

However, for members in most structures and machines the elastic strength of the material is the maximum usable stress for the material. And, in all cases, the working stress is considerable less than the maximum usable strength, the value of the working stress depending on the conditions and uncertainties discussed in Art. 7.

But even though the elastic strength of a material is the maximum usable strength, there are two very important reasons why the value of the ultimate strength of material used in force-resisting members should be known:

(1) In order to give a measure of the reserve strength of the member, which it brought into play in resisting abnormally large overloads due to accident and unforeseen conditions, as for example, a derailment on a bridge, an excessive settlement of a foundation of a building or bridge, a tornado, the running of a testing machine after the specimen has been removed and until the moving head comes in contact with the weighing table, etc. In such cases, although the structure or machine may have its usefulness temporarily destroyed, nevertheless, if the material has a large reserve strength (a large insurance factor against total collapse) much less damage may be done, particularly where loss of life is involved, than if the reserve strength is lacking and hence allowing collapse or rupture to occur. (This topic is discussed also in Art. 143 and 148.)

(2) But, even if the loads on a member do not cause stresses, in the member *as a whole*, that are greater than the elastic strength of the material, there may be localized stresses at various points in the member that are considerably above the yield-point of the material; high localized stresses frequently occur at sudden changes of sections such as at the roots of threads, at the edges of the holes in the plates of a riveted joint, etc.; they are also caused by straining the members during erection in order to make them line up, or by heat treatment or cold working, etc. And, although only a small part of the total volume of the member may be overstressed, still if the material of this small volume does not have a considerable reserve strength it will rupture and may be the initial cause of the rupture of the whole member. In fact stresses (both local and general) greater than the elastic strength of the material are by no means uncommon in many types of structures that are supposedly subjected to relatively low working stresses. The spalling off of paint frequently noticed on steel members in buildings and other structures indicates the great need of "reserve strength," and at the same time suggests that in some classes of work the reserve strength of the material is too frequently and too

confidently relied on to resist secondary stresses not taken into account in proportioning and erecting the structure or machine.

Summary: The ultimate strength is of value in selecting suitable working stresses for brittle material such as cast iron, concrete, rope, canvas, hard drawn copper, etc., since for such materials the ultimate strength, if determined carefully (see Art. 136), is probably the best measure of the maximum usable stress. In structures made of ductile material, however, the elastic strength must, as a rule, be considered to be the maximum usable strength of the material. But, even though the working stress as well as the selection of the material is based largely on the elastic strength, the ultimate strength, particularly in its relation to the elastic strength, is of great importance as a measure of the *reserve strength* of the material; that is, the strength which becomes available in resisting unforeseen overloads, excessive localized and secondary stresses, etc., thereby adding an insurance factor against total rupture or collapse of the structure or machine. The ratio of the elastic strength to the ultimate strength is frequently called the *elastic ratio;* in specifications for structural steel this ratio usually is required to be at least one-half, the elastic strength being measured by the yield-point.

Further, test results show that a value of approximately one-half of the ultimate static strength is a fairly satisfactory measure of the fatigue strength of wrought ferrous materials when the repeated stress is completely reversed, that is, when the stress varies from a tensile to an equal compressive stress (see Art. 155 for further discussion).

QUESTIONS

19. For what kind of a material and under what conditions may the ultimate strength be regarded as the maximum usable strength of a material?

20. What is meant by reserve strength of a material? Why is reserve strength of importance even when the material is used in a structure that would be damaged by stressing the material above the elastic strength. Define elastic ratio.

139. Static Compressive Strength.—Although the preceding discussion has referred mainly to the tensile strength of materials, the statements also apply with few exceptions to the compressive strength of the materials. There are, however, additional facts that should be considered in connection with compressive strength.

For most structural compression members (columns) made of ductile material the yield-point must be regarded as the ultimate strength of the material since the plastic yielding accompanying the yield-point allows the member to bend, after which rupture or collapse may soon follow (see Art. 92). Now, if a structure which contains both tension and compression members such as a building, bridge, crane, etc., and which is made of ductile material (structural steel, etc.), for which the ultimate strength in tension and compression are approximately the same, is to be designed, and if the same working stress is used for compression members as is used for tension members, the resistance of the tension members to total collapse will be greater than that of the compression members. In other words, the reserve strengths of the tension[7] members will be greater than that of the compression members, and since the reserve strength of the whole structure is likely to be that of the compression members it is evident that, *if the reserve strength of the structure is of real importance* (as it is in most structures), the working stresses for the compression members should be lower than for the tension members. This principle is recognized in city building ordinances and other building specifications by specifying lower[8] working stresses for compression members than for tension and flexural members.

Further, the small local yieldings, which lower the proportional limit (see Art. 137*a*) but which in a tension member may not be indicative of structural damage may be the beginning of real structural damage in a column, since local yielding is likely to be followed by bending of the whole column, and hence for columns the measure of the elastic strength of the material needs more careful consideration than for members used in tension or flexure; the conditions (amount of rolling of metals, of moisture in timber, of internal stresses in metals, etc.), which affect the elastic strength should be considered in determining working stresses.

Form of Test Specimen.—In determining the compressive

[7] The reserve strength of large *built-up* tension members is not as large in comparison with compression members as might be expected from the results of small test specimens, due to the unevenness of stress in the component parts of the built-up member. Further, the tensile strength of full-sized solid members may be only about 0.85 of the tensile strength of test specimens. (See Trans Am. Soc. Civ. Eng., Vol. 61, p. 191; also Eng. News, July 5, 1906.)

[8] See footnote 17 of Chap. XI.

strength of a material the specimen should not be long enough to allow it to bend, and yet it should be long enough so that the unevenness of the pressure on the ends of the specimens will not cause noticeable unevenness of stress on cross-sections in the central portions of the specimen, that is, in the gage length. A cylindrical specimen having a length two to four times its diameter is satisfactory, provided the ends are machined smooth in the case of metals or imbedded in plaster of paris if the specimen is concrete, stone, etc. Thus, a cast-iron or steel cylinder 1 in. by 3 in., and a concrete cylinder 6 in. by 12 in. are in common use.

140. Static Shearing Strength.—The shearing ultimate strength of a material may be found by testing the material in direct shear in various ways as indicated in Fig. 257. The shearing elastic strength (proportional limit, Johnson's apparent elastic limit, yield-point, etc.) is not found from a direct shearing test because of the difficulty of measuring the deformation. The shearing elastic strength is found from a torsion test of a cylindrical specimen (for one type of a torsion testing machine see Fig. 258). The stresses on each cross-section of the specimen are shearing stresses and the maximum stress may be calculated from the twisting moment which is applied and measured by the machine (Fig. 258), and the shearing unit-strain may be calculated from the value of the angle of twist; the angle of twist, θ, for a gage length, l, being measured by the scale and pointer attached to the specimen as indicated in Fig. 258. Therefore, a shearing stress-strain diagram may be plotted.

Thus if a gradually increasing twisting moment, T, is applied to a solid cylindrical bar of ductile material, such as structural steel, the stress-strain curve will be as shown in Fig. 259; the shearing unit-stress, s_s, and the shearing unit-strain, ϵ_s, as shown in Art. 26 and 28, are found from the equations

$$s_s=\frac{Tc}{J} \quad \text{and} \quad \epsilon_s=\frac{c\theta}{l}.$$

From this curve the approximate values [9] of the shearing proportional limit (P, Fig. 259), Johnson's apparent elastic limit (E, Fig. 259), yield-point (Y, Fig. 259), etc., may be found by the same methods as were discussed in Art. 137 in connection with a tension test. However, in a torsion test it is usually more con-

[9] See below for reasons why these values are approximate.

venient to draw a torque-angle of twist ($T-\theta$) curve; this curve will have the same form as that given in Fig. 11, since c, J, and l depend only on the dimensions of the specimen which are con-

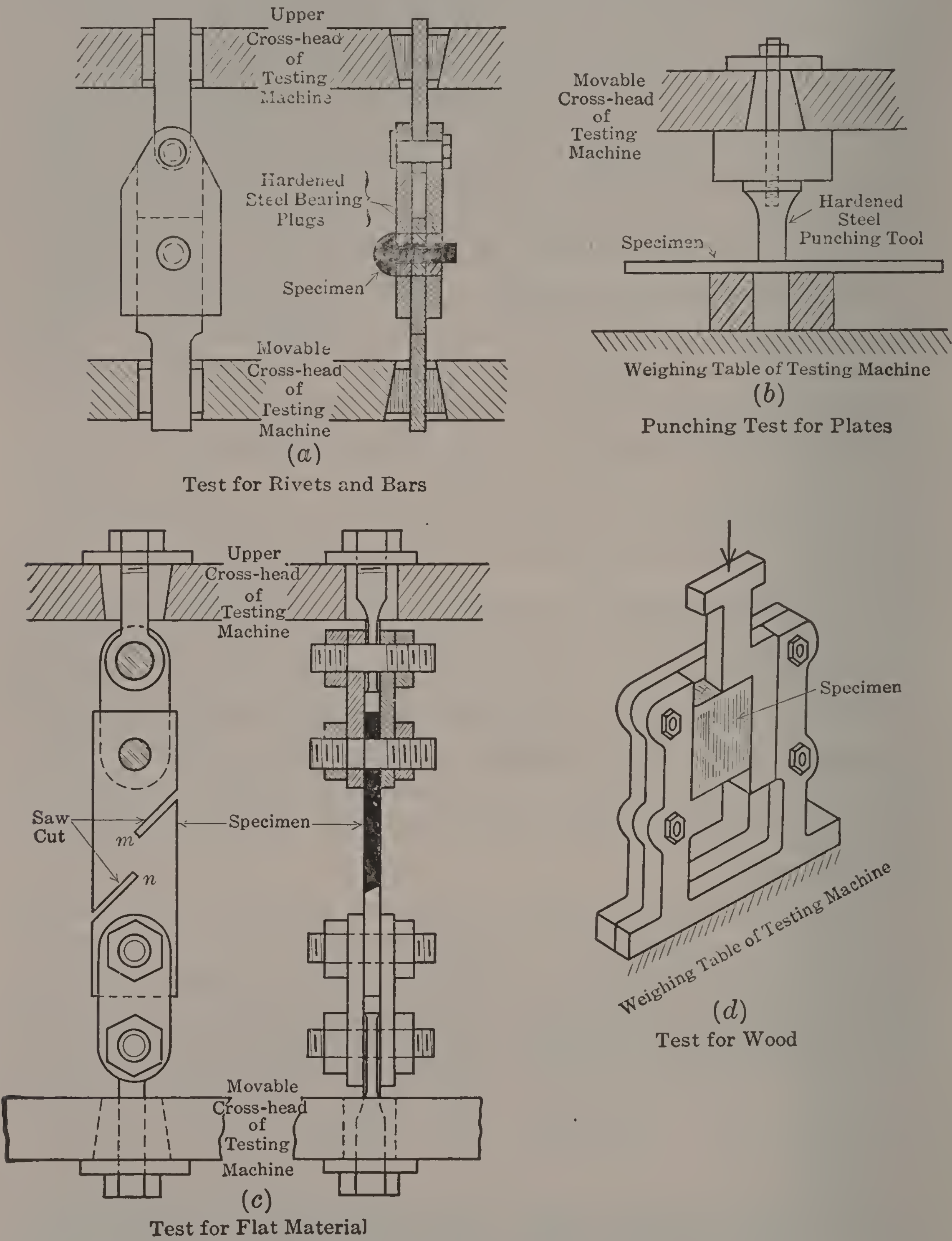

Fig. 257.—Methods of testing material in direct shear.

stant. The torque, T, at the proportional limit or yield-point, etc., may then be found and the corresponding unit-stress calculated from this torque.

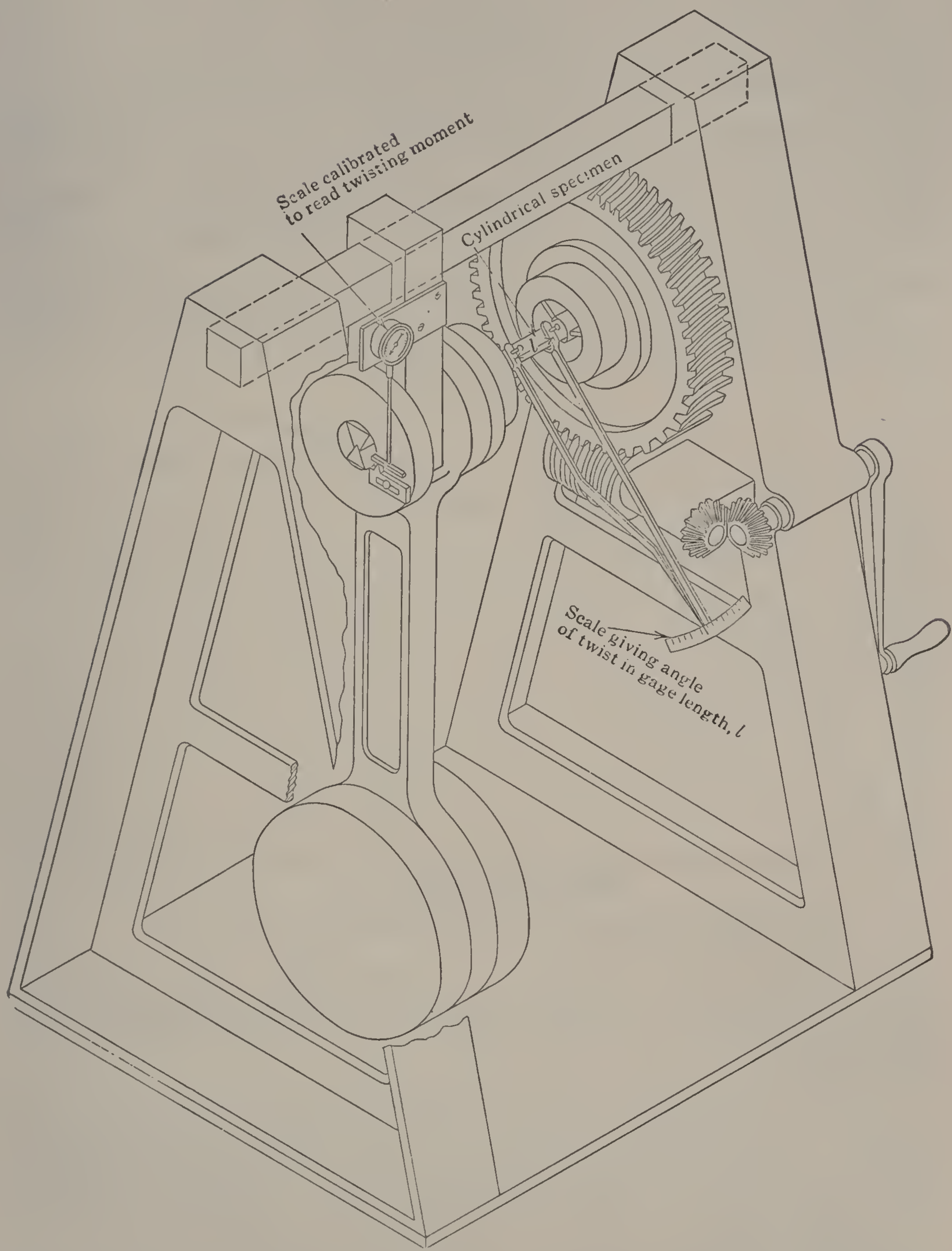

FIG. 258.—Torsion testing machine. By turning the handle the large gear is rotated, thereby exerting a twisting moment on the right-hand end of the specimen. This moment twists the specimen and deflects the pendulum to which the left-hand end of the specimen is attached; the deflection or swing of the pendulum is made to measure the value of the twisting moment.

True Measure of Shearing Elastic Strength.—Since the shearing unit-stress in a bar subjected to torsion, increases directly with the distance from the center of the bar, the fibers at the surface reach their proportional limits first, but the increase in the angle of twist due to the yielding of these surface fibers will be too small to be measured by the strain-measuring apparatus. Hence, the proportional limit, as found from the stress-strain (or torque-angle of twist) curve for a solid bar will not represent the real proportional limit of the material. By using hollow thin-walled cylindrical torsion specimens the proportional limit indicated by

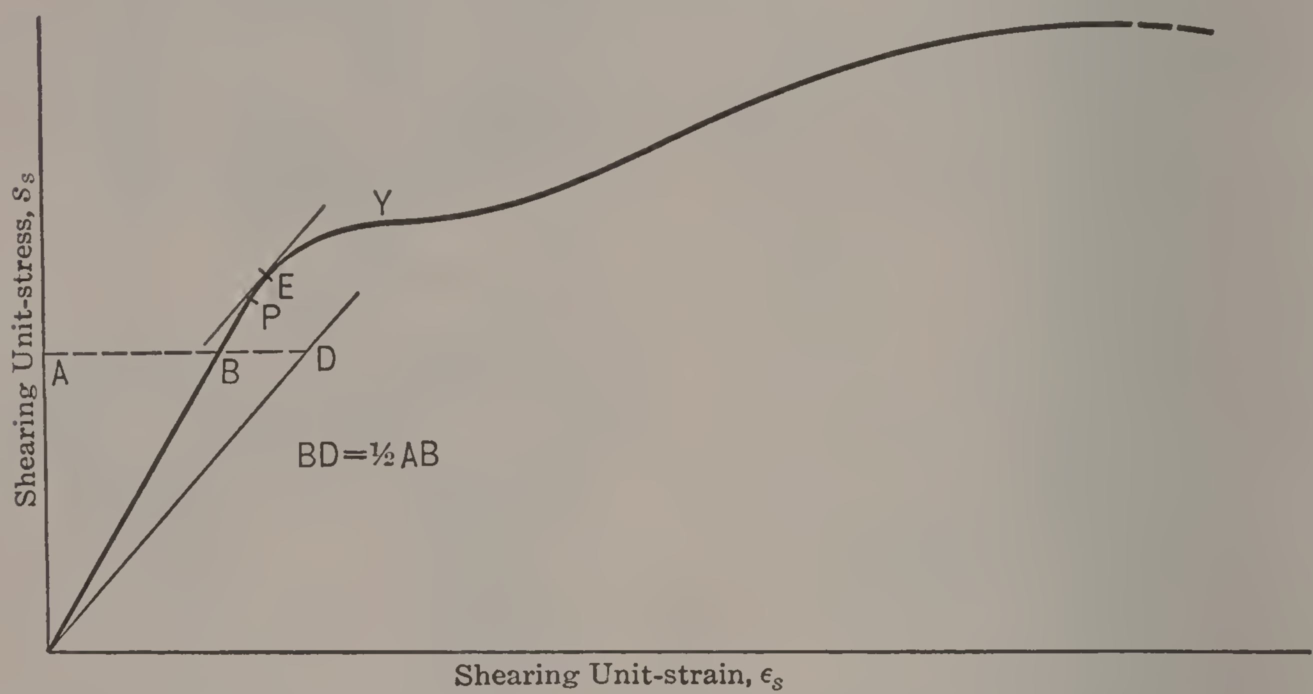

FIG. 259.—Shearing stress-strain diagram for ductile steel.

the curve will be very close to the true value for the material since nearly all the material reaches the proportional limit at the same time.

Tests [10] have shown that the true shearing elastic strength of ductile and semi-ductile steels is about eighty-five hundredths (0.85) of the elastic strength found from a test of solid cylindrical torsion specimen.

Relation between Shearing and Tensile Elastic Strengths.—Tests [10] have shown also that the shearing elastic strength of ductile and semi-ductile steels does not vary much from six-tenths (0.6) of the tensile elastic strength, whether the elastic strengths are measured by the proportional limits, the useful limit points, or the yield-points.

[10] Bulletin No. 115, Engineering Experiment Station, University of Illinois.

QUESTIONS

21. State reasons why the allowable working stresses in columns made of ductile material should, in general, be lower than in tension and flexural members.

22. Why is the shearing elastic strength of a material found by means of a torsion test instead of a direct shearing test?

23. Why is a *hollow* cylindrical torsion specimen required, in order to obtain an accurate value of the shearing proportional limit of a material?

Ductility

141. Definition of Ductility.—Ductility is that property of a material which enables it to acquire large permanent deformation and at the same time develop relatively large stress. It is closely associated with the properties plasticity and malleability since plastic and malleable materials can be worked (hammered, pressed, etc.) into various forms and will retain the forms impressed upon them; they acquire, therefore, large permanent deformations but the resistance that the material offers to the external forces impressed on it while being deformed is either relatively small or of secondary importance. Thus, lead and gold can be hammered into thin sheets and are said to be malleable; putty can be formed into various shapes with very little pressure and is said to be plastic; low carbon steel acquires a relatively large permanent elongation when subjected to a tensile load and maintains its resistance to the load and is, therefore, said to be ductile. There is, however, no sharp line of demarkation between the properties, ductility, plasticity, and malleability, but it is important to note that ductility is a property of a *load-resisting* material.

142. Measure of Ductility.—There is no absolute quantitative measure of ductility. For comparative purposes the most frequently used measures are the values of the percentage of elongation and the percentage of reduction of area of a specimen tested in tension. The percentage of elongation is found by dividing the increase in the gage length after rupture has occurred by the original gage length and multiplying the result by 100. The percentage of reduction of area is found by dividing the difference between the area of the original and ruptured cross-sections by the original cross-section and multiplying the result by 100.

After the maximum load on the specimen of a ductile material has been reached the specimen begins to " neck down " and the load on the specimen decreases as the necking down continues (see Fig. 250). As indicated in Fig. 260, the deformation per unit length is very much greater at the neck-down portion than elsewhere. Therefore, the percentage elongation means little as a measure of ductility unless the gage length over which the deformation is measured is also specified. Thus a material that has a percentage of elongation of 32.5 in 8 in. might have a percentage of elongation of 60 in 2 in. (see Fig. 260).

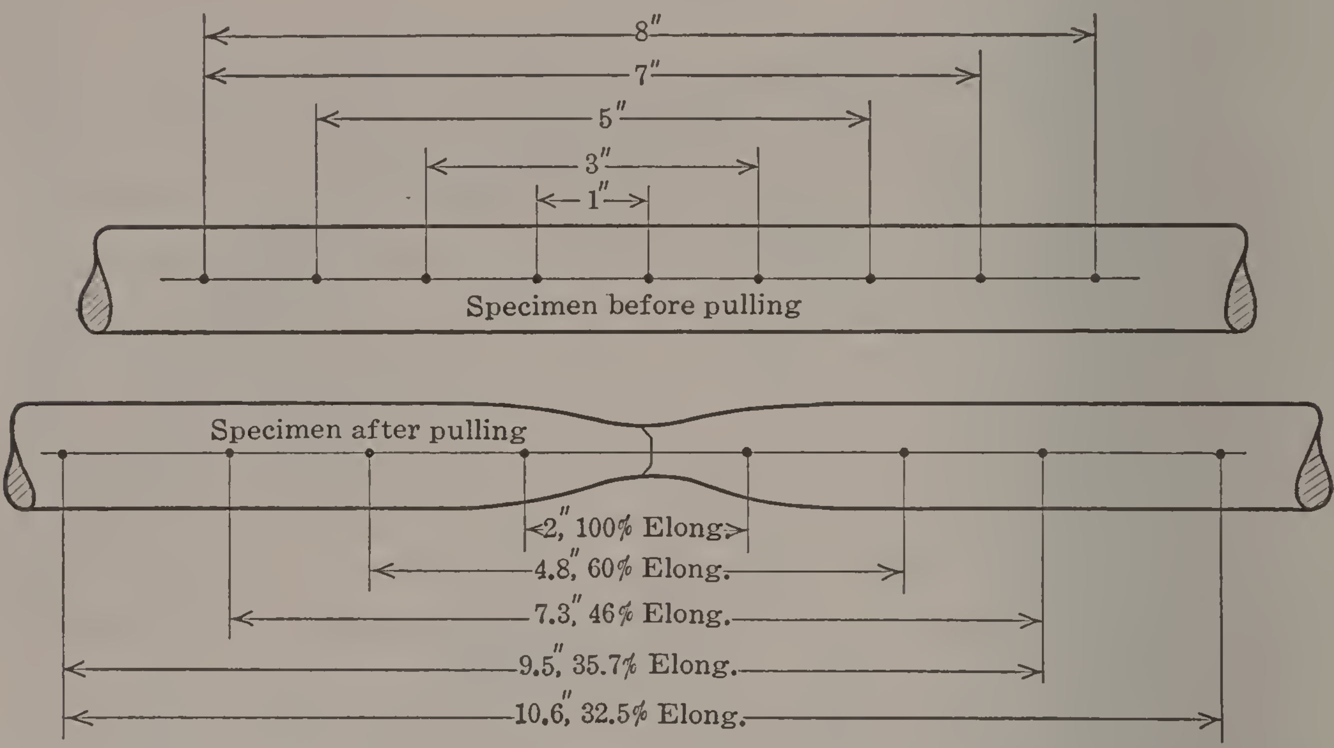

FIG. 260.—The influence of the gage length on the percentage of elongation.

Since during the " necking down " the load falls off rapidly and hence the material is in the process of rupturing, and since " necking down " should never occur in a member in a structure, there is justification for omitting the deformation that occurs during " necking down " in obtaining the percentage of elongation as a measure of ductility. Thus, according to this view, the ductility would be measured by the length OE (in Fig. 250). In commercial testing, however, the added time and expense involved in deducting the elongation due to the necking down has prevented the adoption of this method for measuring ductility. The percentage elongation, therefore, as usually found, includes the deformation due to the necking down (EF, Fig. 250) and also the small amount of elastic deformation (OA', Fig. 250).

Need for Other Methods.—Various grades of steel and of copper alloys differ widely in the way they stretch, some contracting to a narrow neck and some stretching uniformly and breaking with very little necking down; for such material the percentage of reduction of area in addition to the percentage of elongation is sometimes desired. Further, specimens of thin sheets of metal fail with almost no elongation, due to tearing from one side, and the ruptured area of such specimens is difficult to measure, and hence a bend test is frequently used as an indication (rough measure) of their ductility.

This test consists in bending the specimen through 180° without causing cracking of the material on the outside of the bent portion, the radius of the bend being zero for thin (less than $\frac{3}{4}$ in.) specimens of ductile material, and about equal to the thickness of the specimen for thicker specimens.

143. Significance and Need of Ductility.—Values that are considered to be satisfactory for the percentage of elongation and percentage of reduction of area for various materials are determined mainly by the values obtained from material that is known to be approximately the best that can be produced, by the methods in use. In other words, it is practically impossible to state that a material must have a given percentage of elongation in order to be satisfactory for a given use. A value of 15 per cent elongation might be considered very satisfactory for a high carbon steel, whereas 25 per cent might be considered to be unsatisfactory for a low carbon steel, etc. Thus the values of the percentages of elongation and reduction of area are a help in indicating to the engineer whether the material has been manufactured properly, in addition to being a measure of a property of the material.

Why is Ductility Needed in Load-resisting Material?—Is ductility of importance in material of members of structures and machines whose function or usefulness would be destroyed if the deformation of the member as a whole were large enough to make use of the ductility of the material?

The answer to this question is closely connected with the discussion of " reserve strength " in Art. 138 and also with the discussion of " toughness " in Art. 147 and 148. Ductility is needed in the materials of structures and machines for the following reasons:

1. To help prevent the destruction or collapse of the structure

due to excessive overloads, particularly impact loads, caused by accident, etc. This will be discussed more in detail under "toughness," Art. 147.

2. To relieve localized stresses in members by allowing the material to yield locally; this redistributes the stress, without causing any appreciable deformation of the member as a whole; thus the yielding prevents local failure of the material which in many cases would be the initial failure leading to the collapse of the whole member. These local stresses may be due to:

(*a*) Loading that is different from that assumed in the design, and to secondary stresses not considered in the design; for example, the distribution of the load to the rivets in a riveted joint in design is usually assumed to be uniform, whereas the loads on the rivets are known to vary widely; the settlements of foundations frequently cause large changes in the loads on the members in the superstructure; the localized or secondary stresses (not considered in the design) in the eye of a steel eye-bar due to the bending action in the eye is sometimes sufficient to cause flaking off of the paint, etc.

(*b*) Abrupt changes of section, such as corners in crankshafts, roots of threads, key-ways, etc., at which high stresses always exist.

(*c*) Non-homogeneity of the material due to a segregation in the ingot, and non-metallic inclusions and other defects in the case of steel; knots in timber; etc.

(*d*) Straining members of structures during fabricating and erecting.

Ductility, therefore, is of great importance in materials of most structures and machines and the amount of ductility is usually made as great as possible consistent with adequate strength; for, in general, the ductility of steel and other metals is low if the strength is high.

Values of percentage of elongation for various materials are given in the tables of Chapter XVI.

QUESTIONS

24. Define ductility. What quantities are used as a measure of ductility?

25. Why should the gage length be specified in giving the percentage of elongation?

26. Why is ductility desired in a force-resisting member when, in order to make use of this property in the member as a whole, the member must take large permanent deformations and hence fail to fulfill its function in the structure?

27. State several causes of localized yielding in members of structures, and give illustrations.

28. Is there any way of deciding how much ductility is required in material for a given use?

Stiffness

144. Definition and Measure.—The property of stiffness of a material is the rate at which the stress in the material increases with the strain. Hence, the stiffness of a material, corresponding to any unit-stress, may be measured by the slope of a line drawn tangent to the stress-strain curve at the point representing that unit-stress.

The stiffness of steel, or any other material that has a straight stress-strain diagram up to the proportional limit, is constant at all stresses below the proportional limit and is measured by the modulus of elasticity of the material $\left(E=\frac{s}{\epsilon}, \text{Art. } 5\right)$, that is, by the rate of increase in unit-stress with unit-strain. Stiffness, therefore, is measured in the same units as is unit-stress (pounds per square inch) since ϵ is the ratio of a length to a length and hence is merely a number.

As shown in the tensile stress-strain diagram of mild steel in Fig. 250, the stiffness of mild steel when subjected to a tensile stress represented by H (about half-way between the yield-point and the ultimate strength) is very much less than at stresses below the proportional limit. At the yield-point the stiffness is zero. Further, it will be noted that Johnson's apparent elastic limit (Art. 137*d*) is the unit-stress at which the stiffness of the material is 50 per cent less than at stresses below the proportional limit.

Since in most structures the primary stresses developed are less than the elastic strength of the material the modulus of elasticity measures the stiffness of the material as used in such service, and hence is the most important measure of stiffness *for design purposes*. However, the low value of the stiffness of the material at points of high localized stress may be of importance, as for example, in the resistance of columns in which the stiffness of the material is an important factor and in which localized stresses have a relatively large influence.

For material that has a curved stress-strain diagram (Figs. 261 and 262*a*), the modulus of elasticity is frequently taken as the slope of the tangent to the curve at zero stress, and is called the *tangent* or *initial* modulus. Thus, in Fig. 262(*a*) the slope of the line *OA* is the tangent modulus. But the tangent modulus does

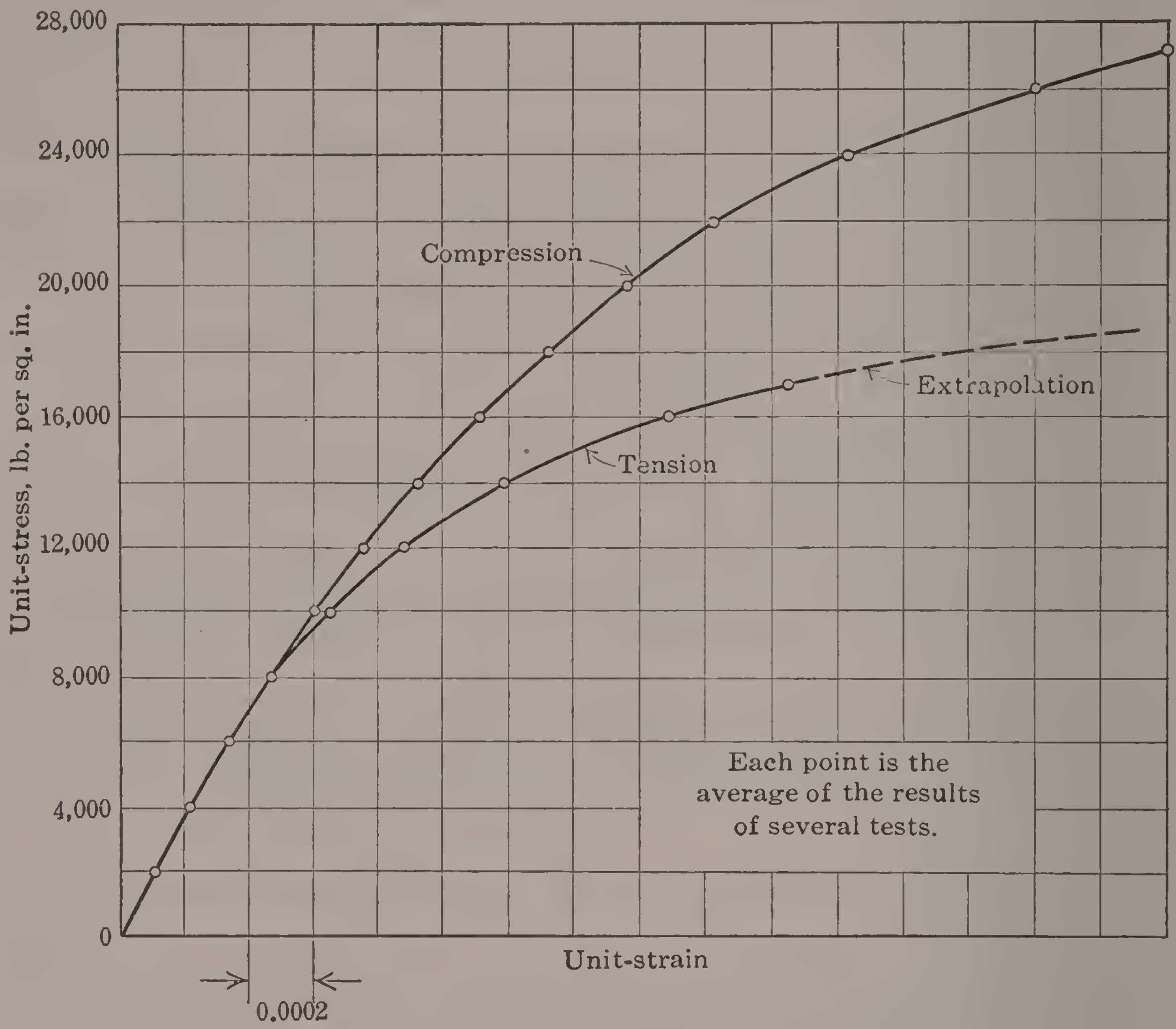

Fig. 261.—Stress-strain diagrams for cast iron.

not measure the stiffness of the material when the material is resisting the working stress; this value, however, is frequently needed as, for example, in the design of reinforced concrete columns which involves the ratio of the moduli of elasticity of steel and concrete at the working stresses. The modulus of elasticity, therefore, is sometimes taken as the slope of a line *OBC* where the ordinate to *B* represents the working unit-stress; the slope of *OBC* is called the *secant modulus*. Although the secant modulus does not, according to the definition of stiffness, measure the stiffness of the material at the stress *B′B*, it is approximately an average value of the stiffness as the material is stressed up to the working stress.

In Fig. 263 are shown the stress-strain curves for four structural materials plotted to the same pair of axes and to the same scales. These curves show that at stresses below the elastic strengths of the materials, wood is only about one-twentieth as stiff as steel, and that cast iron is about one-half as stiff as steel. The medium carbon steel has the greatest static strength

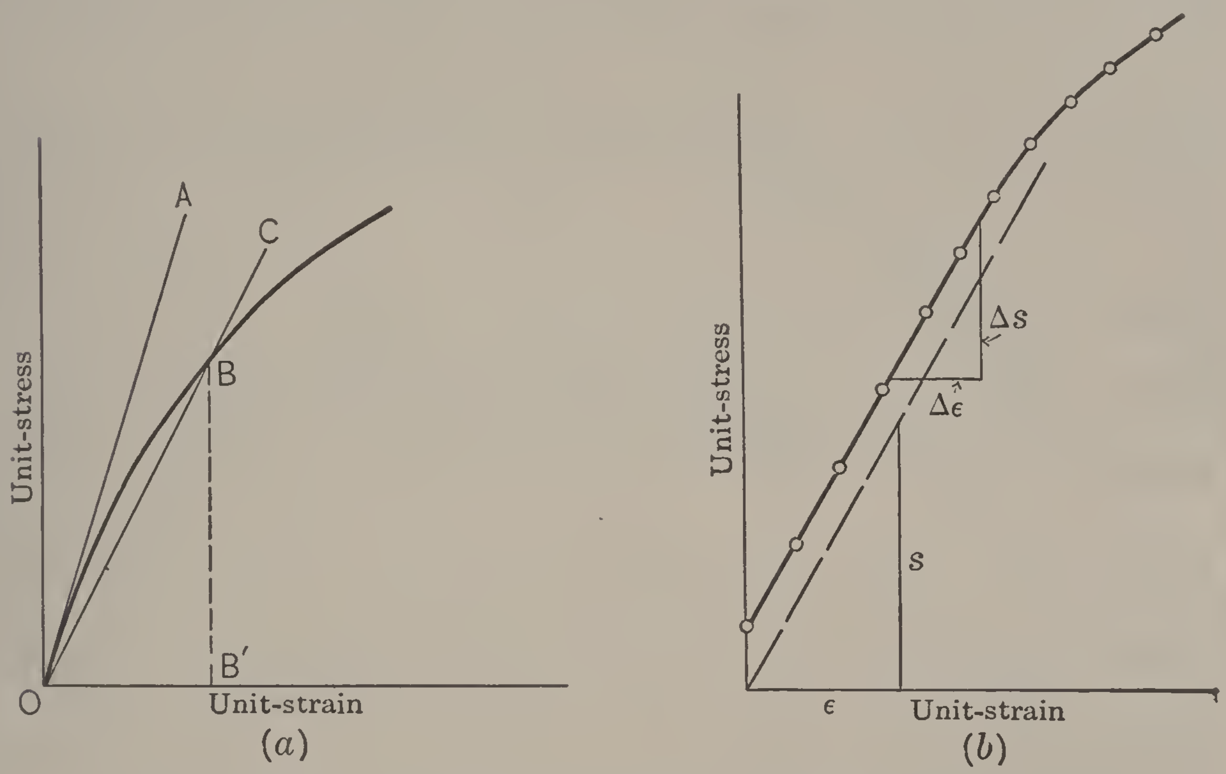

Fig. 262.—Methods of obtaining modulus of elasticity.

and the low carbon steel is the most ductile, but both grades of steel have the same stiffness at stresses below the proportional limits. In fact the tensile moduli of elasticity of practically all steels including heat-treated alloy-steels are approximately the same; namely, 30,000,000 lb. per sq. in., although the elastic strengths may vary from 25,000 to 200,000 lb. per sq. in. The compressive modulus of elasticity is nearly the same as the tensile but the shearing modulus of elasticity of steel (and of most metals) is much less than the tensile or compressive modulus. In other words, the shearing strain in steel increases with shearing stress much faster than elongation increases with tensile stress.

In plotting a stress-strain curve from test data it frequently happens that the curve does not pass through the origin; the slope of the curve, which represents the modulus of elasticity, is not then expressed by the ratio $\frac{s}{\epsilon}$. To obtain the modulus of elasticity a line may be drawn through the origin and parallel to the plotted curve (Fig. 262*b*) and the value of $\frac{s}{\epsilon}$ as obtained from this line will be the modulus of elasticity; or, the difference between two values of the unit-stress may be divided by the difference in the corresponding values of the unit-deformation $\left(E = \frac{\Delta s}{\Delta \epsilon}\right.$, Fig. 262*b*). The scale of plotting should be such as to cause the curve

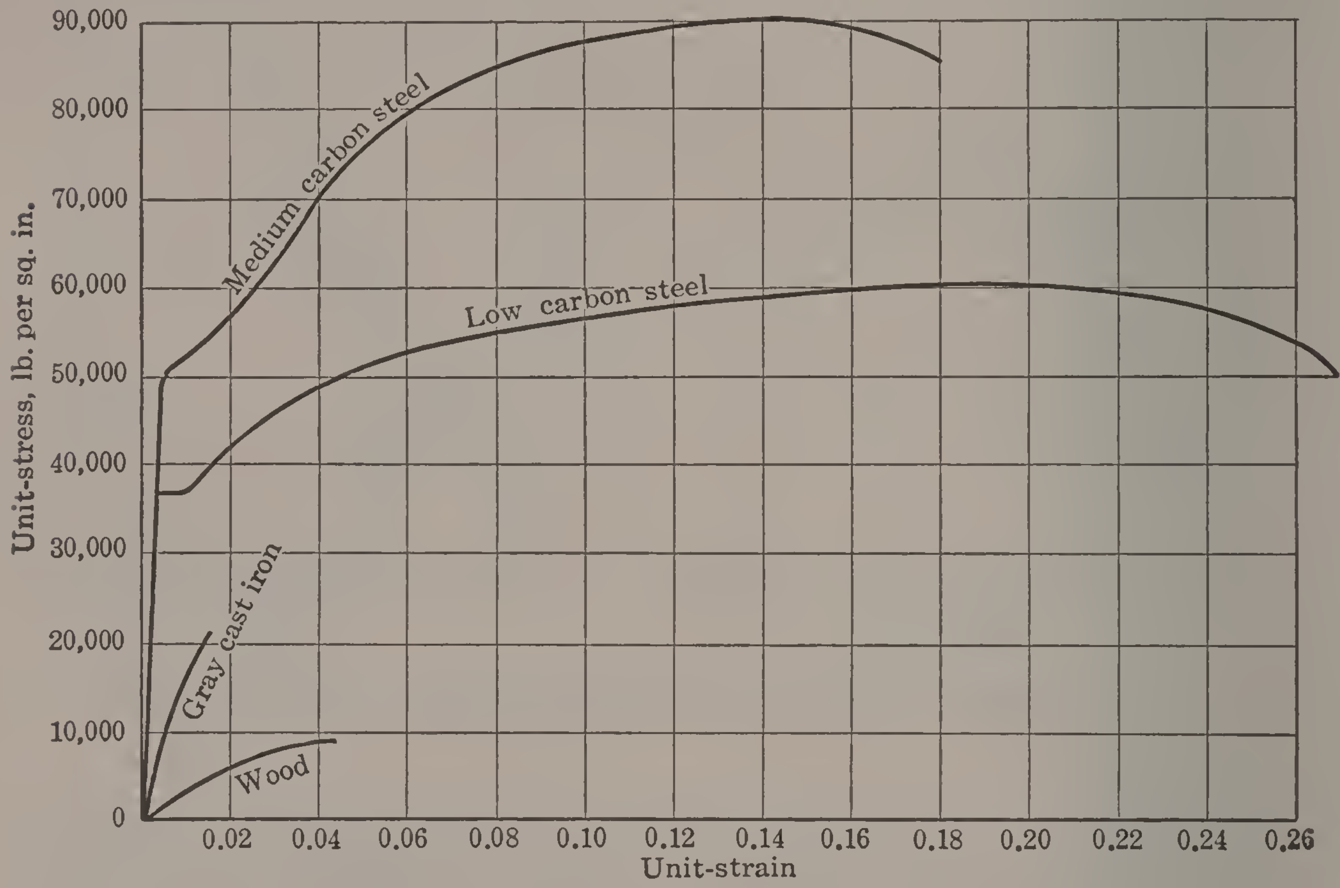

FIG. 263.—Stress-strain diagrams for four structural materials.

to make an angle of from 30° to 45° with the vertical in order to obtain a reliable value for the modulus. Further, the longer the gage length the more accurate will be the value of the modulus; a gage length of not less than 8 in. is recommended by the American Society for Testing Materials. The extensometer used should measure the deformation on at least two sides of the specimen, and the same precautions concerning axial loads should be taken as in determining the proportional limit (see Art. 137*a*). Values of moduli of elasticity for various materials are given in the tables of Chapter XVI.

145. Use and Significance of Stiffness.—A distinction should be made between the stiffness of a structural *member*, such as an eye bar, a beam, or a shaft, etc., when subjected to loads and the stiffness of the *material* of which the member is made. The stiffness of the member depends on the size and form of the member as well as on the stiffness of the material. The manner in which the size and form of a member affects its stiffness is discussed in Part I and, as there pointed out, stiffness of a member rather than its strength may be the governing factor in a design. Under these conditions the stiffness of the *material* is an important property of the material. Thus a machine tool such as a planer lathe tool, drill press, or grinding machine may deflect too much for accurate work even though the stress in it is relatively small; a long shaft may twist so much that troublesome vibrations are set up under fluctuating loads even though the shaft is amply strong; a long floor beam may deflect so much that the plastered ceiling beneath will crack even though the maximum stress in the beam is well within the elastic strength of the material; the guides for a locomotive cross-head may deflect enough to interfere with the running of the engine, etc.

If the members in the above illustrations were made of steel the trouble could not be relieved by merely substituting a member made of a stronger grade of steel; for, if the loads and dimensions remain the same the stresses do not change, and hence the deflection of the member could change only because of a change in the stiffness of the material, but the stiffness of the strongest (high carbon) steel is the same as that of the softest (low carbon) steel, consequently the deformation or deflection of the member would not change.

If, however, the floor beam in the above illustration were made of wood, the beam could be made much stiffer by replacing it by one having the same dimensions but made of steel, for although the stress in the steel beam would be the same as that in the timber beam (the loads and dimensions remaining constant), the deflection of the steel beam would be much less.

The property of stiffness is also of great importance in determining the resistance of material to energy loads, that is, in determining the amount of energy the material can absorb; for, as shown in Art. 116 and discussed in the following article, the amount of energy that a material can absorb per unit volume (cubic inch) is $\frac{1}{2}\frac{s_e^2}{E}$ when stressed to its elastic strength, s_e. Thus if the stiffness of a material is small (assuming the strength to remain constant) the capacity of the material to absorb energy without being stressed above its elastic strength is large. Thus the low stiffness of wood makes it desirable for railway ties and for spokes in automobile wheels, etc.

Values of the moduli of elasticity of various materials are given in the tables of Chapter XVI.

QUESTIONS

29. Define stiffness. How is the stiffness of a material found? In what units is it expressed?

30. Is the stiffness of structural steel when stressed above the yield-point greater or less than when stressed below the yield-point?

31. How is the stiffness of a material measured if the material has a curved stress-strain diagram? Define tangent modulus and secant modulus.

32. Compare the stiffness of steel, wood and cast-iron at stresses below their elastic strengths.

33. Is the stiffness of steel at stresses below the proportional limit the same for all grades of steel? Compare the tensile and shearing stiffness of steel at stresses below the proportional limits.

34. Under what conditions is it desirable for a force-resisting member to be made of material having a high degree of stiffness? Having a low degree of stiffness?

Resilience; Elastic Energy Strength

146. Definition, Measure and Significance of Resilience.—Resilience is the property of a material that enables the material to give up or release energy (that is, to do work) as the stress is released.

The energy recovered when the stress is released from the elastic limit is called *elastic resilience*, and the energy recovered *per unit volume* is called the *modulus of elastic resilience* of a material. Now all the work done in stressing a material to its elastic limit (or to any stress less than the elastic limit) is stored in the material and can be recovered as the stress is released; that is, none of the energy is dissipated in heat in causing structural damage, and hence, the work done per unit-volume in stressing the material to the elastic limit is equal numerically to the modulus of elastic resilience, k. Therefore, k is represented by the area OAB (Fig. 264) and is equal to

$$k=\frac{1}{2}\frac{s_e^2}{E},$$

as explained in Art. 116.

But the elastic energy strength of a material is the amount of work that can be done on (or absorbed by) the material per unit volume in stressing the material to its elastic limit (see Art. 116); therefore, *the modulus of resilience is a measure of the elastic energy strength of the material.* The elastic energy strength of a *structural member*, however, depends on the form and dimensions of the member as well as on the elastic energy strength of the *material* of which the member is made (see Chapter XIII).

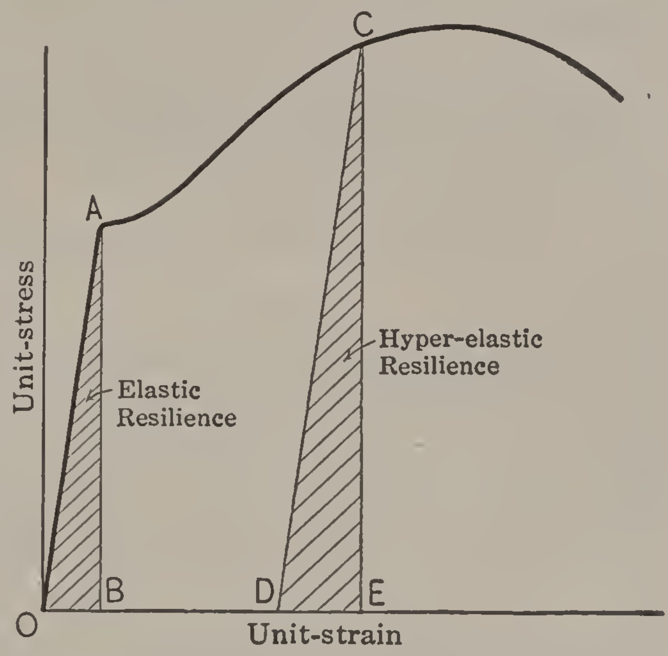

FIG. 264.—Areas representing resilience.

In obtaining the *tensile* modulus of resilience, s_e and E in the above expression are the tensile elastic limit and the tensile modulus of elasticity, respectively; whereas, in expressing the shearing modulus of resilience s_e and E are the shearing elastic limit and the shearing modulus of elasticity, respectively.

The ideal material, then, for resisting energy loads that must not cause plastic deformation in the material is one having a high elastic strength and a low modulus of elasticity. Thus, the modulus of resilience of high carbon and alloy steels is much larger than that of low carbon steel due solely to the greater elastic strength, since the modulus of elasticity is not influenced by the carbon or alloy content. Further, the shearing modulus of resilience of steel, k_s, is approximately equal to the tensile modulus, k_t. This is shown as follows:

$$k_s=\frac{1}{2}\frac{(s_e)_s^2}{E_s} \quad \text{and} \quad k_t=\frac{1}{2}\frac{(s_e)_t^2}{E_t},$$

but from Art. 5 and 140, $s_s=0.6s_t$ and $E_s=\frac{2}{5}E_t$.

Therefore,

$$k_s=\frac{1}{2}\frac{(0.6)^2}{0.4}\frac{(s_e)_t{}^2}{E_t}=\frac{1}{2}\frac{(s_e)_t{}^2}{E_t}, \text{ approximately.}$$

Thus, the resistance that steel can offer to energy loads without acquiring plastic deformation is approximately independent of the kind of stress to which the steel is subjected (see Art. 113).

On the other hand, bronze is as strong as some grades of steel and its stiffness (modulus of elasticity) is considerably lower. Therefore bronze may be nearly as good material for a spring, etc., as the poorer grades of spring steel. Likewise timber is low in stiffness and hence possesses a fair degree of resilience even though its elastic strength is low. (See Table XI of Chapter XVI for a comparison of the elastic energy strengths of various materials).

If a bar is stressed beyond the yield-point of the material most of the work done is dissipated in causing structural damage. If, however, the stress is released from some value such as EC (Fig. 264), the stress-strain diagram will take the form CD. That is, the part DE of the total unit-strain OE is elastic deformation, and the area DCE represents the energy recovered per unit-volume and hence measures the modulus of resilience corresponding to the stress EC. But it is convenient to use the term *modulus of elastic resilience* for the energy represented by the area OAB and to use the term *modulus of hyper-elastic resilience* for the energy recovered per unit volume when the stress in the material is released from a value above the elastic strength of the material.

The modulus of hyper-elastic resilience is a property, however, of little use in selecting material for resisting loads, whereas the modulus of elastic resilience is of great importance in selecting materials for members to be subjected to energy loads that are to develop stresses less than the elastic strength of the material, such as springs, connecting rods of forging hammers and of rock drills, various automobile parts, etc.

As shown in Table XI, the value of the modulus of elastic resilience of various grades of steel vary from about 15 to 670 in.-lb. per cu. in., the latter value being obtained from special alloy steel used for springs, axles, gears, etc., which must resist energy loads without being permanently distorted.

QUESTIONS

35. Define resilience, elastic resilience, modulus of elastic resilience. What is the quantitative expression for the modulus of elastic resilience?

36. Define elastic energy strength of a material and show that it is measured by the modulus of elastic resilience.

37. What use of material requires a high modulus of elastic resilience?

38. What is the modulus of hyper-elastic resilience? Is it of importance in force-resisting members?

39. Compare the tensile and shearing moduli of elastic resilience of steel.

Toughness; Ultimate Energy Strength

147. Definition and Measure of Toughness.—Toughness is that property of a material that enables it to absorb energy while being stressed above its elastic strength, that is, while incurring plastic deformation. Thus, the more work that has to be done in stressing a material from its elastic limit to its ultimate strength the greater is the toughness of the material.

Measure of Toughness.—One measure of the toughness of a material is the amount of work per unit-volume of the material required to rupture the material when subjected to a gradually increasing (static) load. This measure of toughness is represented by the area under a stress-strain curve; this area, however, includes the work done in stressing the material to the proportional limit (that is, the resilience of the material), but the area under the straight-line part of the stress-strain diagram of *ductile* materials is negligible in comparison with the total area as is evident from an inspection of Fig. 250(a); and it can be neglected even for relatively brittle materials without

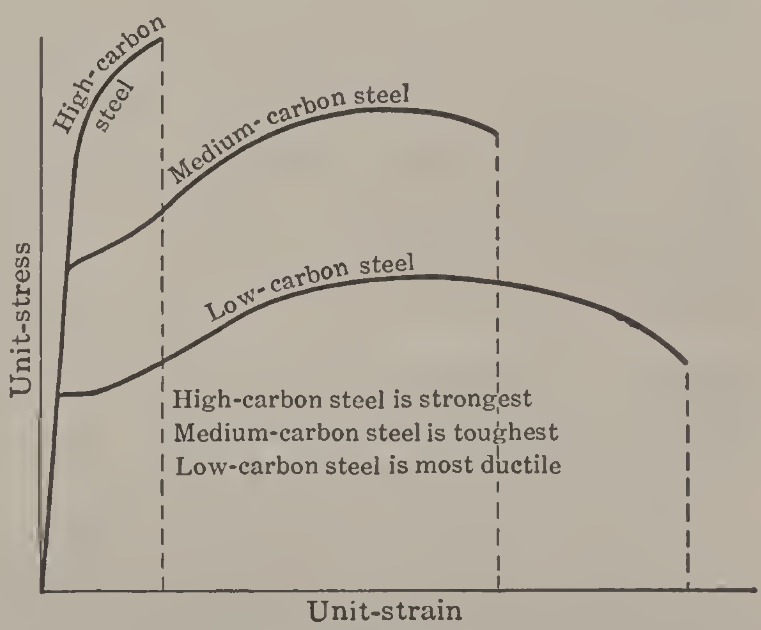

Fig. 265.—Toughness represented by areas under stress-strain curves.

serious error particularly since an exact quantitative measure of toughness is not as a rule required.

The stress-strain diagrams in Fig. 265 show that the high carbon steel has the greatest static strength (both elastic and ultimate) and the greatest modulus of resilience (or elastic energy strength), that the medium carbon steel has the greatest toughness and that the low carbon steel has the greatest ductility.

Since the area under the stress-strain diagram of a *ductile* material is roughly proportional to the product of the ultimate static strength and the percentage of elongation of the material, this product, sometimes called the *toughness index number* or *merit number*, is used frequently for comparative purposes as an approximate measure of toughness. (See Art. 116 for expressions from which an approximate value of the area under a stress-strain curve may be found.) Values for toughness of several materials are given in Table XI of Chapter XVI.

148. Significance of Toughness.—When a *static* overload due to accident or other cause is applied to a member of a structure, resulting in stresses greater than the elastic strength of the material, the ultimate strength of the material determines whether the member will successfully resist the load or will rupture, that is, the reserve strength of the material (see Art. 138), if sufficient, becomes available in preventing rupture. On the other hand, if an energy overload, that is, an amount of energy in excess of the resilience of the member (see Art. 114), is applied to the member by a moving body, thereby causing stresses in the member greater than the elastic strength of the material, the toughness of the material determines whether the energy of the moving body will be absorbed by the member or will rupture the member. In other words, the toughness of a material is a measure of the ultimate energy strength of the material and it furnishes reserve strength for resisting excessive energy loads.

In service, then, in which material is subjected to energy loads and in which the stresses are to be kept within the elastic strength of the material without provision for accidental overloads, etc., the resilience, $\frac{1}{2}\frac{s_e^2}{E}$, of the material is of prime importance. Thus the material in piston rods on steam forging hammers and rock drills should have high resilience rather than toughness; likewise, the materials in most springs and many gears are selected because of their high resilience. On the other hand, the material in the frame of a railway locomotive or car which in service is practically certain to be subjected to impact loads that will cause stresses above the yield-point of the material, toughness in addition to resilience is of importance. The same is true to a slightly lesser extent of the material in ships and bridges. For example, the stresses in the members of bridges under conditions of service are expected to be less than the static elastic strength of the material (except the localized stresses, see Art. 143) but if a minor derailment or other accident should occur on a bridge, requiring that a considerable amount of energy present in the moving train shall be absorbed by the bridge the material must possess toughness if the bridge does not collapse. Automobile frames, axles, gears, etc., furnish other examples of members for which toughness, in addition to resilience, is an essential property.

149. The Single-blow Notched-bar Impact Test.—This test was devised in response to a demand for a quick and convenient method of determining the suitability of a material for resisting impact and energy loads. It was urged that the resistance of a material to impact should be determined from a test in which the load is applied by a moving body. It is doubtful, however, whether this test determines the ultimate impact or energy resistance (toughness) of a material *as the material is used in engineering structures and machines*, for reasons discussed below.

The test is usually made with a Charpy (or Izod [11]) pendulum machine (Fig. 266); the specimen (Fig. 267) is a notched rectangular beam. The energy lost by the pendulum in rupturing the specimen is the value found from the test; this value is sometimes referred to as the " Charpy value."

In order to cause a specimen of ductile material to rupture when struck by the pendulum, instead of merely bending, the specimen must be notched.[12] The form of the notch has a marked

[11] The Izod machine is used mainly in France and England and is practically the same as the Charpy machine except that the specimen used in the Izod machine is tested as a cantilever beam whereas the Charpy machine tests the specimen as a simple beam; the form of the notch also is usually somewhat different.

[12] Although the controlling idea in devising the test was that of applying the load with impact, it is now recognized that the significant factor in the test is the notched form of specimen and the accompanying concentration of stress at the root of the notch.

influence on the amount of energy required to rupture the specimen and hence the values found from the tests are meaningless even for comparative purposes unless the specimen is standardized; two forms of specimens frequently used are shown in Fig. 267.

Now, the portion of the material in the specimen that absorbs the greater part of the energy is near the bottom of the notch but

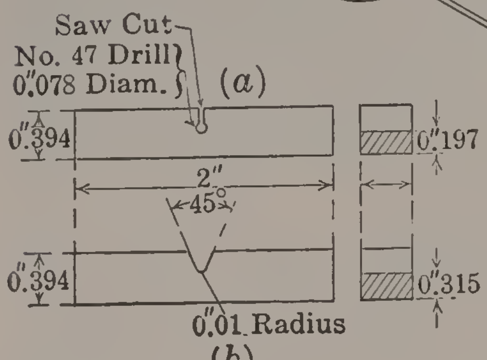

FIG. 267.—Forms of test specimens.

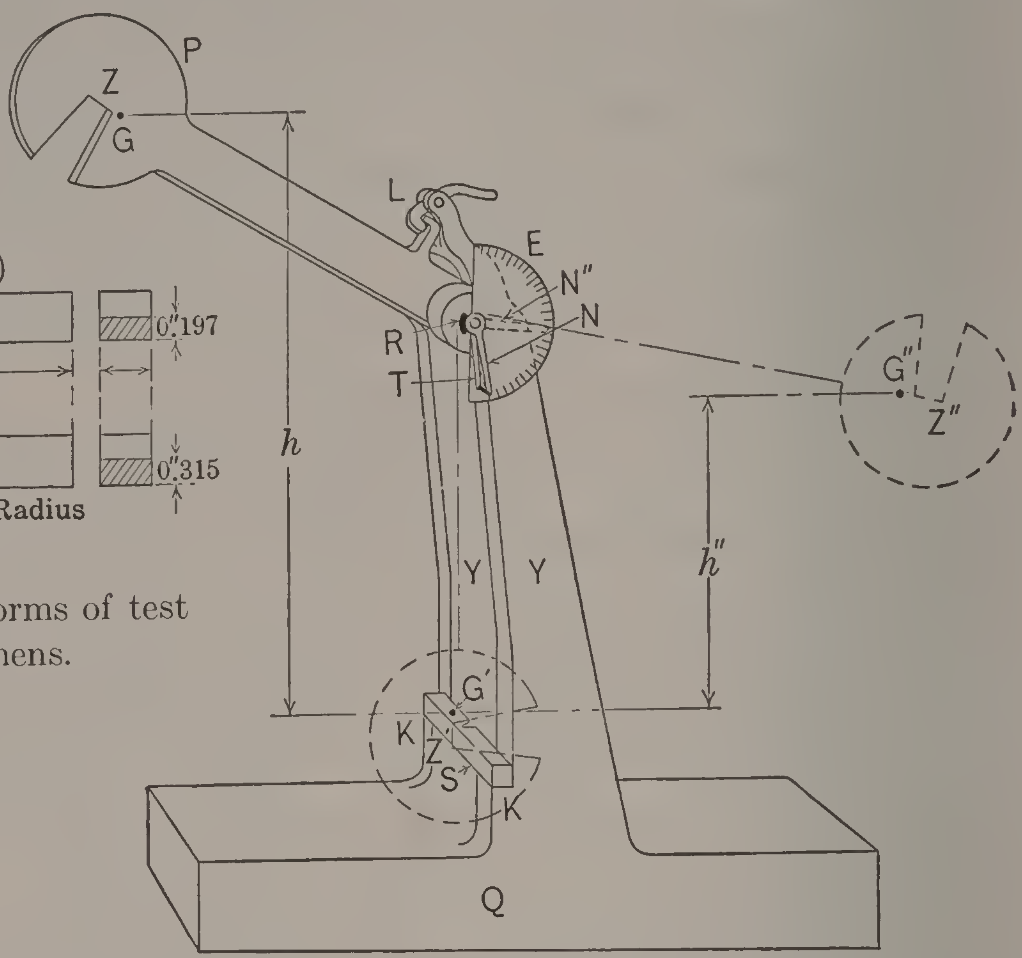

FIG. 266.—Single-blow, notched-bar impact testing machine. (The Charpy machine.) The pendulum when released from its position Z strikes the specimen S and rises to the position Z''. Thus the center of gravity, G, of the pendulum lowers a distance $(h-h'')$ as the pointer N moves to the position N''; the pointer remains at N'' after the pendulum swings back from its position Z'' and hence the angle of rise is indicated on the scale E. The energy expended by the pendulum in rupturing the specimen is calculated from the measured quantities and is the value obtained from the test.

the amount of material involved is indefinite and indeterminate, and hence the toughness, or energy absorbed per unit volume, is indeterminate. Again, it is doubtful if the resistance of the material, as found in this test, is the resistance offered by a material to an impact or energy load as such loads occur in service; the high local stress at the root of the notch, where the section is abruptly reduced, requires that the material at the root of this notched section shall fail suddenly before other parts of the specimen can be stressed, and hence before those parts can be made to absorb much of the energy. This action produces sudden tearing of the material, and materials in machines and structures probably seldom if ever are subjected to such severe action, even though some machine members have rather abrupt changes of sections, such as at sections at roots of screw threads, tool marks and deep scratches, at roots of small radii, etc. (Some engineers, nevertheless, feel that the test is not too severe.)

The test, however, is rather widely used and is of special value in determining whether certain heat treatments of steel have been carried out successfully since heat treatments that have been poorly done affect the value obtained from the single-blow notched-bar test very markedly. This fact is of great importance since much of the material used in automobiles, aeroplanes, and many special machines are heat treated in order to secure high strength and light weight and they must also have considerable toughness or reserve impact strength to resist accidental impact overloads. By heat treating the proper steels these desirable properties can be produced provided the heat treatments are successfully carried out, and hence a convenient test that sharply differentiates between successful and poor heat treatments is of great value. But materials should not be selected on the basis of Charpy values alone; the results of impact tests should be interpreted in connection with other properties of the material.

QUESTIONS

40. Define toughness of a material. How is it measured?

41. What is meant by toughness index number or merit number?

42. What uses of material require a high degree of toughness? Give illustrations.

43. Does the single-blow notched-bar impact test measure the toughness of the material? If not, why? What significance has this test?

Hardness

150. Definition and Measure of Hardness.—The meaning of hardness of a material is different with the different operations or services in which the material is used; the operations in which hardness is of special importance being scratching, abrasion, cutting, and penetration or indentation. Hardness seldom if ever signifies a single physical property of a material, but it is, however, always associated with stresses that are accompanied by plastic or permanent deformations. Hardness, therefore, may be defined as the resistance which a material offers to the complex and indefinite stresses that are brought into action in operation involving scratching, abrasion, cutting and penetration.

The particular hardness that is of greatest importance in connection with the material in load-resisting members is the resistance to penetration or indentation. Two methods of measuring the resistance of a material to indentation are in wide use; namely, the Brinell ball test, which is a slow or static penetration test, and the Shore scleroscope test, which is a rapid or dynamic penetration test.

151. The Brinell Ball Test.—The Brinell machine (Fig. 268) applies a pressure of 3000 kilograms (or 500 kilograms for very soft material) for fifteen to thirty seconds to a ball 10 millimeters in diameter, and this ball, which rests on the surface of the test specimen, causes a permanent indentation in the specimen, the diameter or width of the indentation being measured by a simply and conveniently constructed microscope. The load in kilograms is divided by the surface of indentation in square millimeters and the resulting average intensity of pressure is called a *Brinell number*. (Brinell numbers for various materials are given in Table I, Chapter XVI.)

152. The Shore Scleroscope.—The Shore scleroscope (Fig. 269), releases a small steel hammer from a height of about 10 inches, allowing it to fall freely in a glass tube with a diameter slightly larger than that of the hammer. The steel hammer weighs $\frac{1}{12}$ ounce and has a diamond striking-point the face of which is rounded to a definite radius. When the hammer strikes the specimen it penetrates the surface of the specimen before rebounding and hence produces a permanent but minute deformation. In so doing, part of the energy acquired by the hammer in falling is absorbed, and hence the height of rebound is less than the original height of fall. The rebound of the hammer is observed on the vertical scale and this reading (called the scleroscope number) is taken as the measure of the hardness of the material; the harder the material the higher the rebound. The scale reading of 100 is frequently fixed as the height of rebound of the hammer when it strikes a special hardened steel surface; this surface is then called the standard surface.

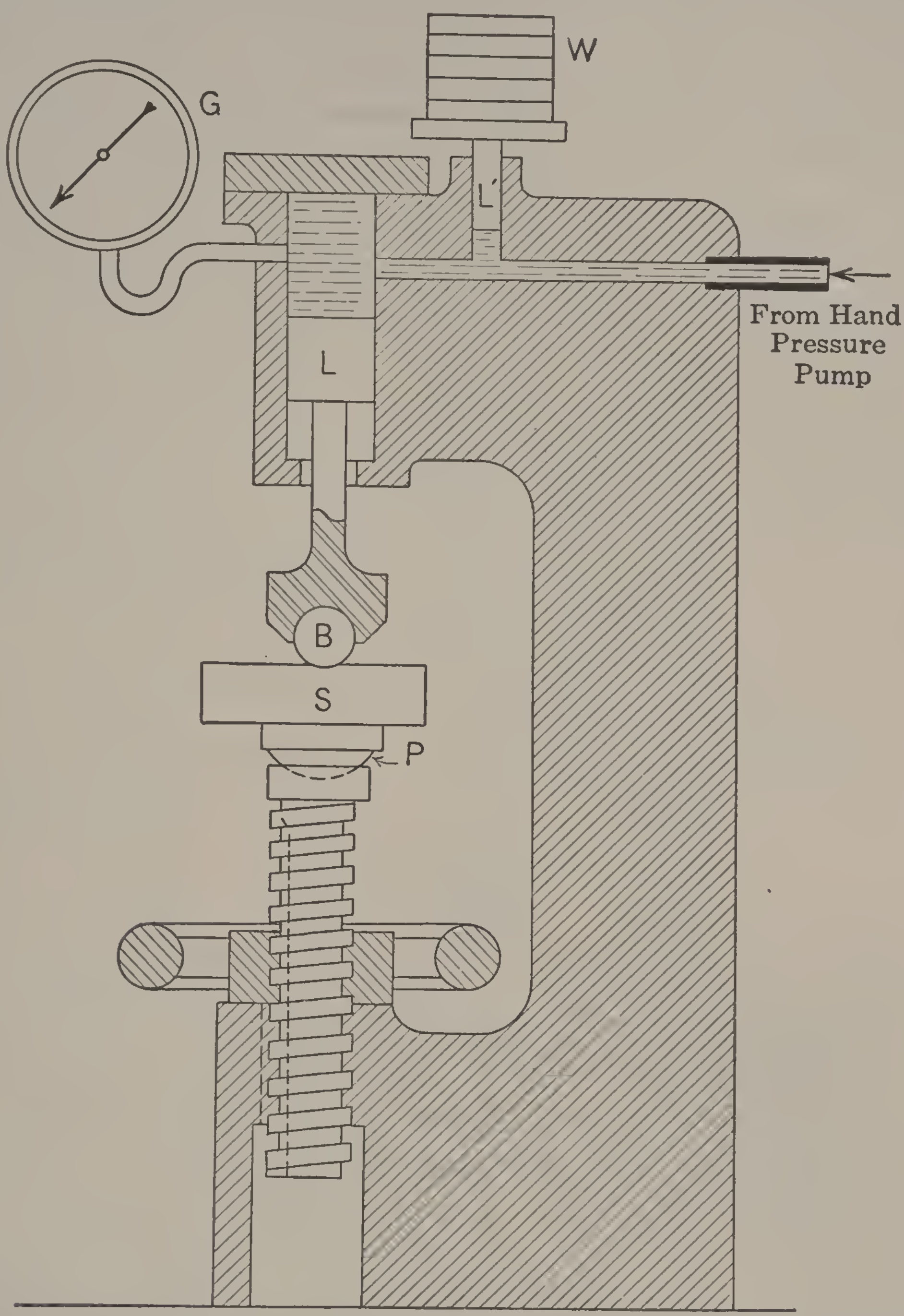

FIG. 268.—Brinell hardness testing machine. The ball B is pressed into the specimen S by the oil pressure on the piston L. The pressure is maintained constant for about thirty seconds by pumping (by hand) so that the known weight W is kept in a floating position. The specimen rests on a spherical-seated bearing block. The diameter of the indentation indicates the hardness of the material.

153. Limitations of the Tests.—The Brinell test is not satisfactory for testing extremely hard materials, since the ball itself deforms too much, nor is it satisfactory for thin sheets of material such as saw blades, etc. Further, in testing finished products whose surfaces must not be dented, the Brinell test cannot be used.[13] In these three respects the scleroscope is to be preferred, but, in general, the Brinell test is usually considered to be the more reliable, since in the Brinell test more of the material is tested and

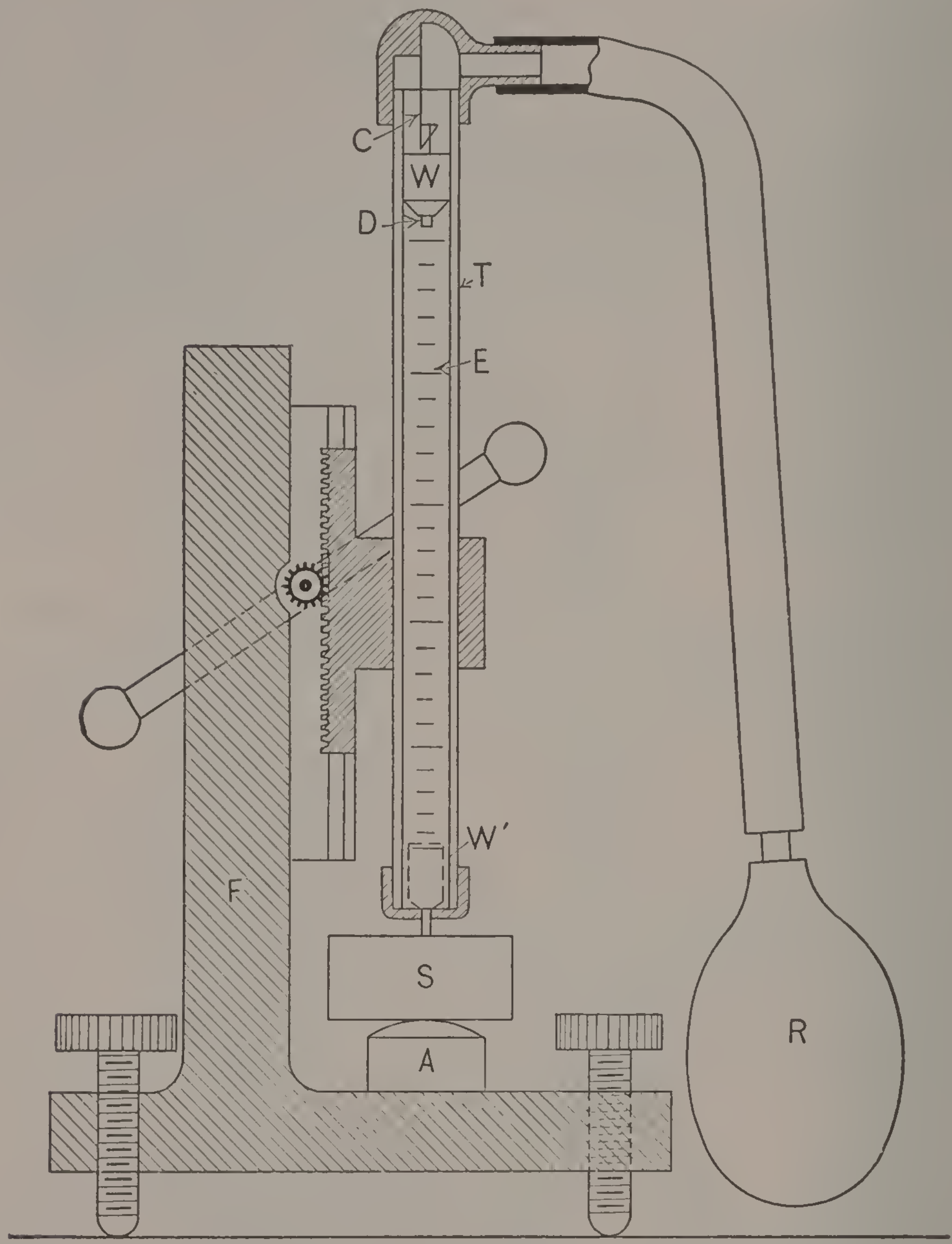

FIG. 269.—Scleroscope hardness-testing machine. The hammer *W* with its diamond *D* falls freely in the glass tube *T*. The diamond point strikes the specimen *S* and rebounds; the height of rebound is read on the scale *E*, the value read on the scale being called the "scleroscope hardness number." By squeezing the rubber bulb *R*, the hammer is raised; the hammer is then held by the latch *C* until the bulb is again squeezed when the latch is released and the hammer falls.

the character or condition of the material and of the surface affects the results less than in the scleroscope test. Thus rubber and wood

[13] A smaller machine called the "Baby Brinell," which uses a ball only $\frac{1}{16}$ inch in diameter, may be used with relatively thin material.

may give the same rebound with the scleroscope, although the two materials differ in hardness.

154. Need and Significance of Hardness.—Since hardness is closely associated with the resisting stresses that accompany plastic deformation, the yield-point or ultimate strength gives a general index of the hardness of a material. In fact, test results [14] show that there is a fairly definite linear relation between the ultimate tensile strength of carbon steel and the Brinell and scleroscope numbers. Therefore, the Brinell and scleroscope tests form convenient methods of estimating the tensile strength of machined or other parts from which it is impossible or not feasible to cut a test specimen.

R. R. Abbott found that the relations between the tensile ultimate strength, s_u, and the Brinell number, B, and scleroscope number s, for carbon and alloy steels was expressed approximately by the following equations:

$$s_u = 0.70B - 26 \qquad s_u = 4.0S - 15,$$

and that the relation between the Brinell and scleroscope number was approximately

$$B = 5.5S + 28.$$

But, as noted above, there is no single test that satisfactorily measures all of the various kinds of hardness required in service. Further, the degree of hardness desired in producing any object has been determined almost entirely by experience; the Brinell, scleroscope, or other hardness tests assist in attaining uniformity of product but not in determining the degree of hardness the material should have for the service required.

Many structural and machine members require a relatively high degree of hardness in addition to some one or more other properties such as resilience, toughness, etc.; thus railway rails, locomotive and car wheel tires, automobile gears and axles, armor plate, cutting edges of steam shovels, jaws of stone crushers, dies for wire drawing, etc., must have high resistance to indentation or abrasion or both. And since strength and hardness of metals are, in general, dependent properties, steel used for the above-

[14] Abbott, R. R. Proceedings, A.S.T.M., Vol. 15, 1915, p. 43 (see also Jour. of Iron and Steel Inst. 1909).

mentioned parts is usually relatively high carbon steel since the strength and hardness of steel increase with the carbon content. But the ductility and toughness of steel decrease with the carbon content and hence for those members which require toughness in addition to hardness special alloy steels are used which possess a fair degree of ductility and toughness as well as high strength and hardness. Thus automobile gears and axles are frequently made of chrome nickel steel, armor plate of chromium steel, jaws of crushers of chromium or manganese steel, etc. And as noted above, after experience has demonstrated the kind of material to be used for a given service and the treatment to be given the material, a hardness test is frequently a convenient way of determining whether the desired properties are being obtained.

Resistance to Wear.—Sometimes the property of hardness as defined above is made to include "resistance to wear." However, resistance to wear depends on the ductility (or plasticity) of the material perhaps more than on its strength-hardness. That is, a good wearing metal must resist the displacement of its particles (and hence have hardness) but when they are displaced it must resist the removal of the particles from the body (and hence have plasticity). In general, if two metals have the same tensile strength the one having the greater ductility will possess the greater resistance to wear, although the resistance to wear of both metals would be increased by increasing the tensile strengths and the fineness of crystalline structure, assuming the ductility to remain constant.

QUESTIONS

44. Define hardness of a material.

45. Describe the Brinell ball test and the Shore scleroscope test. What are the advantages and disadvantages of each?

46. Can either of these tests be used to determine the hardness needed for a given use? If not, how is this determined?

47. Give examples of uses of material in which hardness is one of the essential properties of the material.

Fatigue Strength

155. Definition and Measure of Fatigue Strength.—The fatigue strength of a material is the greatest resistance the material can offer, without rupturing, when subjected to a load that is repeated a great many times. As discussed in Chapter XIV, steel may rupture when subjected to a repeated load even when the *calculated* stress in the material is considerably less than the static proportional limit or elastic limit of the material. In other words, the *fatigue ultimate* strength of the material may be not only less than the static ultimate strength but even less than the *static elastic* strength of the material. Further, a member that fails due to a repeated load seldom gives warning of the approaching rupture; that is, no *measurable* plastic yielding of the member occurs at a stress less than the stress at which rupture occurs. Thus, there is no measurable *elastic* fatigue strength as there is with static strength; the only measurable fatigue strength is the ultimate or rupture fatigue strength. (See Art. 126 for an explanation of a fatigue failure.)

The measure of the fatigue (ultimate) strength of a material is the greatest unit-stress in the material that can be repeated an indefinitely large number of times without causing the material to rupture. This unit-stress is called the *endurance limit* of the material (see Art. 125). Experimental investigations[15] have established, so far, endurance limits only for wrought ferrous metals (rolled and forged steel and wrought iron); comparatively little work has been done on cast-iron or steel castings or on nonferrous metals.

Values of the endurance limits (fatigue strengths) of various grades of iron and steel with completely reversed bending stress are given in Table 7 of Chaper XIV; and in Table XII of Chapter XVI; and the use of these values in obtaining the endurance limits with direct axial stress, with shearing stress and with various ranges of stress are discussed in Chapter XIV. The main object of this section is to discuss various methods of determining the endurance limit of materials.

156. Methods of Determining the Endurance Limit.—(*a*) *Direct Method.* The direct method of determining the endurance limit of a material is to test several specimens of the material as described in Art. 125, and to plot an $s-N$ diagram (see Figs. 237 and 238). Repeated stress tests have not yet been standard-

[15] See Bulletins 124, 136 and 142 of the Engineering Experiment Station of the University of Illinois; an extensive bibliography is given in Bulletins 124 and 142.

ized, but two types of repeated stress testing machines commonly used for determining the endurance limit will here be described briefly.

In Fig. 270 is shown a machine which applies repeated flexural stress by means of a crank and connecting rod and measures the bending moment applied to the specimen, by means of the compression of calibrated springs. Power is furnished by a motor *M* (or from a line shaft) and a crank *C* with an adjustable throw is driven by the motor. The crank is attached to a connecting rod *R* which bends a specimen *S* back and forth. The motion of the specimen is resisted by springs *G* acting through a bent lever *A*. The amount of bending moment applied to the specimen may be varied by changing the throw of the crank and is measured by the amount of compression of the springs *G*. This compression is indicated by the throw of the arm *I* to the end of which is attached a pencil that records the throw on paper wrapped round the drum *D*. The drum *D* is rotated by a worm and wheel drive from the main shaft of the machine, and there is consequently recorded on the paper a diagram whose width is a measure of the bending moment on the specimen and whose length is a measure of the number of applications of stress to the specimen. The number of applications of stress is also indicated by a counter *K*. From the bending moment the maximum unit-stress applied to the specimen can be determined. Usually this type of machine is used to pro-

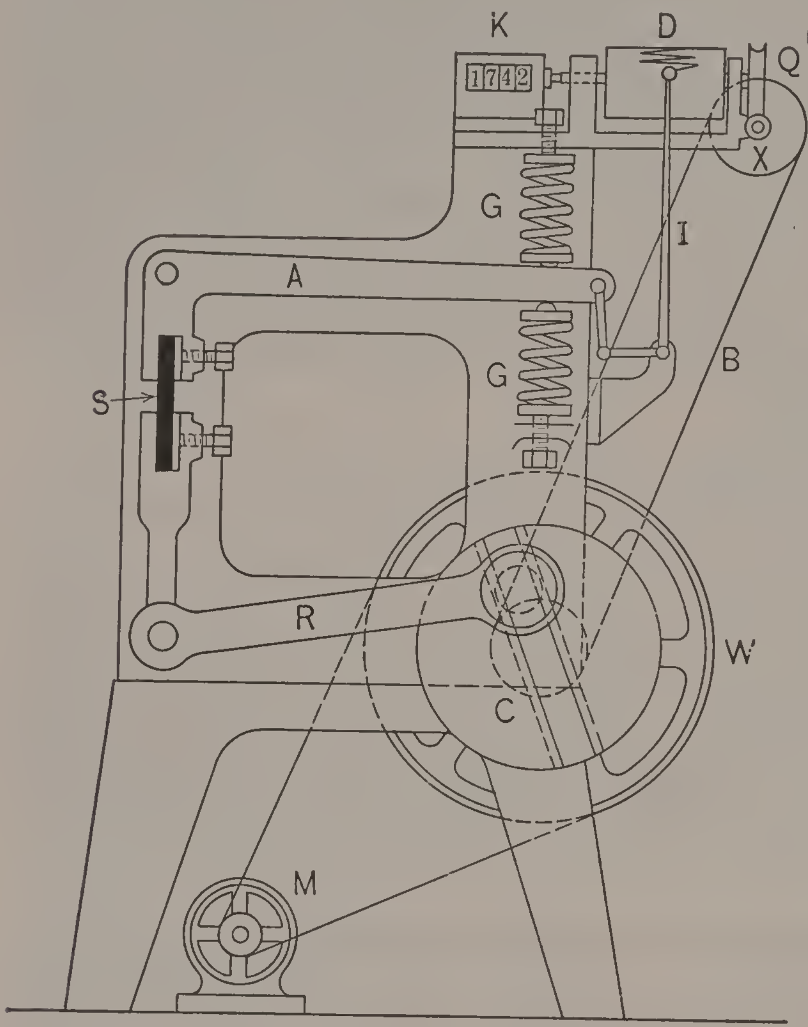

Fig. 270.—Upton-Lewis repeated-stress testing machine.

duce reversals of bending stress, but by varying the springs G other stress ranges can be applied to the specimen. The Upton-Lewis machine (Fig. 270) is the commonest example of this type of machine used in the United States.

Fig. 271 shows in diagram a testing machine which produces reversals of bending stress by the use of a rotating flexure specimen. The specimen S is supported on ball bearings B and driven by a pulley P. Weights are hung from a second set of ball bearings B', and these weights cause bending stresses in the specimen. The bending stress in the upper fibers of the specimen is compression, and in the lower fibers, tension. As the specimen is rotated the

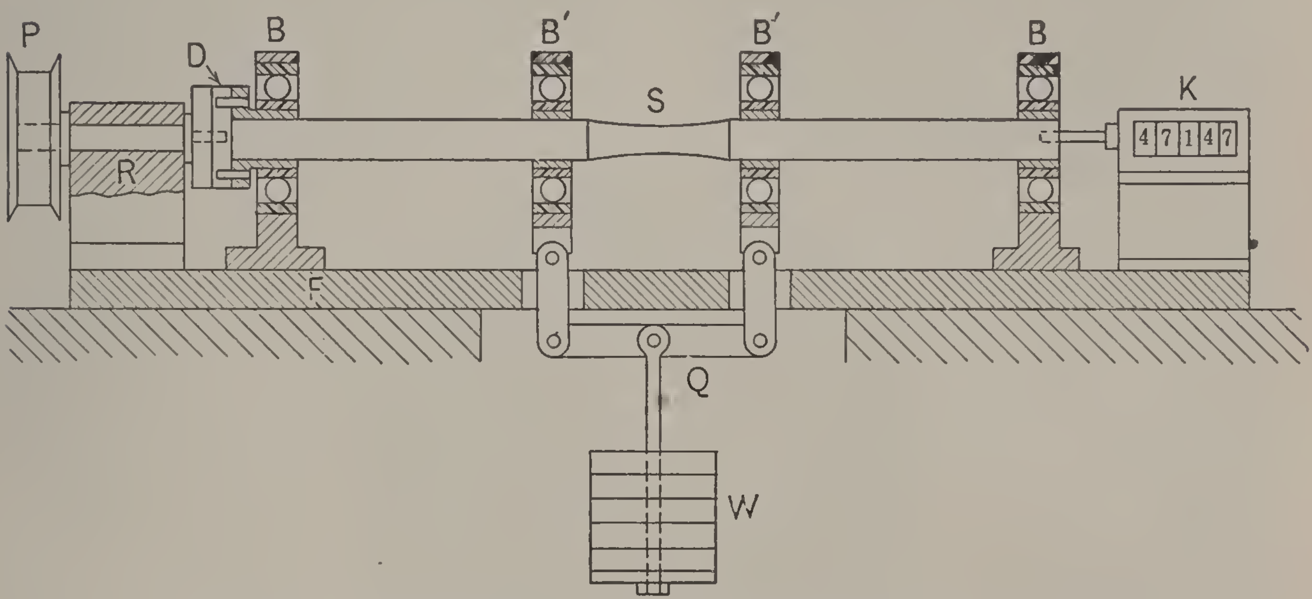

FIG. 271.—Rotating-beam repeated-stress testing machine.

stress for any fiber changes from compression to tension, and the stress is completely reversed. The maximum unit-stress for both tension and compression can be computed from the amount of weights W applied, the dimensions of the specimen, and the distances between bearings; the bending moment is constant for all sections between the two center bearings. A counter K indicates the number of reversals of stress given to the specimen, and when the specimen breaks the counter automatically stops.

(*b*) *Rise-of-Temperature Test.*—The determination of the endurance limit by the actual application of repeated stress as outlined above requires so much time that it is not as serviceable in *commercial* testing as is desired, and hence various short-time or accelerated tests for determining the endurance limit have been tried; one such test that has proven to be fairly reliable for wrought ferrous metals is the rise-of-temperature test.[16] This test consists in subjecting a specimen to a small number of reversals (about 500) of a known stress and measuring the rise in temperature of the specimen, at the section subjected to the maximum stress, by means of a thermo-couple and a galvanometer, or by some other method; the same test is then repeated two or three times, increasing the stress in each successive test until a well-marked rise of temperature can be detected. The endurance limit is taken from a stress-temperature diagram as the unit-stress at which the rate of rise of temperature shows a marked increase. The values of the endurance limit for wrought ferrous metals found by this method vary but little from those found by the direct long-time repeated-stress tests. However, this test, or any other short-time test thus far developed, is essentially supplementary to, rather than a substitute for, the long-time endurance test.

(*c*) *Relation of Endurance Limit to Other Physical Properties.*—It was formerly thought that the elastic limit (or proportional limit) was a measure of the resistance of a material to repeated stress. Now, as pointed out in Art. 137, the elastic limit is an index of the beginning of appreciable plastic action in a material although it is rather an uncertain quantity as discussed in Art. 137, whereas the failure of a material due to repeated stress, as shown by microscope examination of specimens, seems to be a progressive tearing apart, or shearing apart (rupturing) across minute areas of the metal (see Art. 126). Evidence of the truth of this statement is found by examining the photomicrographs in Figs. 272 (*b*) and (*c*) which show the surface of a piece of steel stressed slightly beyond the yield-point when subjected to a single steady load. In Fig. 272(*b*) may be seen a few slip lines, which, under repeated loading, might develop into a fatigue failure similar to that shown in Fig. 273(*b*); there may also be seen in Fig. 272(*b*) dark patches which locate " valleys " caused by the plastic wrinkling of the surface of the steel; this wrinkling is more pronounced in Fig. 272(*c*). The beginning of this plastic action coincides approximately with the proportional limit of the steel rather than with the ultimate strength, and spread of the wrinkled areas is in contrast with the gradual spread of minute fractures which constitutes fatigue failure as indicated in Fig. 273(*b*). Thus it does not appear strange that

[16] See Bulletin 124 of the Engineering Experiment Station of the University of Illinois.

the results of repeated stress tests show a fairly good correlation between the endurance limit and the ultimate tensile strength of iron or steel, whereas the correlation between the proportional limit (or yield-point) and the endurance limit is not so close.

a b c

FIG. 272.—Photomicrographs of steel subjected to a static load: (*a*) Specimen in unstressed state. (*b*) Specimen stressed slightly beyond the yield-point. (*c*) Specimen stressed considerably beyond the yield-point.

Experimental data (see Table XII of Chapter XVI) are rather consistent in showing that, for a rather wide range of steels, the endurance limit *with completely reversed bending stress* is approximately 0.5 of the tensile ultimate stress[17] (nearly all the steels

[17] As pointed out in Chapter XIV, the endurance limit with complete reversals of axial stress (tension-compression) may be as low as 0.60 of the endurance limit with reversal of bending stress. With reversed shearing stress the endurance limit is approximately 0.55 of the endurance limit with reversed bending stress.

tested give values between 0.4 and 0.6). If actual test data from repeated-stress tests are available the endurance limit should be determined from such test data, but if such data are not available it is recommended that the endurance limit for a wrought ferrous metal with completely reversed bending stress be estimated to be 0.45 *per cent of the static ultimate tensile strength.* Further, since the "Brinell hardness number" (Art. 151) is a rough index of the tensile ultimate strength it is also a rough index of the fatigue strength of the material.

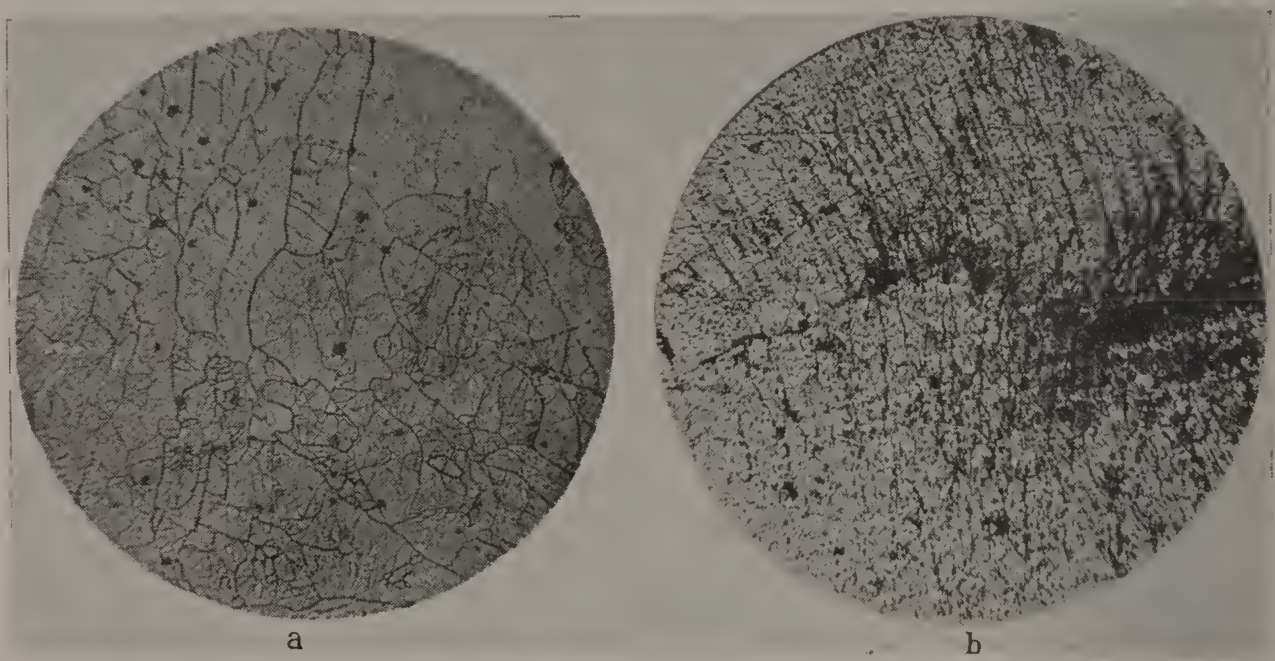

Fig. 273.—Photomicrograph of steel subjected to a repeated load. (*a*) Specimen in unstressed state. (*b*) Specimen after being subjected to many repetitions of stress considerably above the endurance limit.

No values can be given for estimating the fatigue strength of non-ferrous metals or of non-metallic materials until more test data are available.

QUESTIONS

48. Define fatigue strength of a material. What is the measure of fatigue strength?

49. Is there any experimental evidence of an elastic fatigue strength?

50. Describe three methods of obtaining the endurance limit of a material from tests.

51. What evidence is there that the static ultimate strength is a better criterion of the fatigues trength of a material than is the static elastic limit or yield-point?

CHAPTER XVI

TABLES OF PROPERTIES OF MATERIALS

By H. F. Moore[1] and F. B. Seely

157. Use of Tables.—In this chapter are grouped, for convenience, tables giving average values of the properties of the more common structural materials. In using these tables it should be remembered that the values given are average values, and that the values of the properties of any one material may vary considerably from those given in these tables. Therefore, the values given should not be used blindly; the uniformity and general quality of the material should be considered in addition to the properties. If possible, structures and machines in which the material has been used should be studied. When average values of the properties of a material are the only information available the values used in design should err on the side of safety.

[1] Research Professor of Engineering Materials, University of Illinois.

TABLE I

Average Values for Static Strength, Stiffness and Ductility of Iron and Steel

The values given in this table are based on test results from various laboratories.

Material	Strength in Tension Lb. per Sq. In.		Strength in Compression Lb. per Sq. In.		Strength in Shear Lb. per Sq. In.		Modulus of Elasticity Lb. per Sq. In.		Elongation in 2 Inches Per Cent	Brinell Hardness Number
	Proportional Elastic Limit	Ultimate	Proportional Elastic Limit	Ultimate	Proportional Elastic Limit	Ultimate	Tension	Shear		
Gray cast iron	(a)	20,000	26,000	75,000	(b)	(b)	15,000,000	6,000,000	Slight	(c)
"Semi steel"	14,000	35,000	50,000	130,000	(b)	(b)	(c)	(c)	Slight	(c)
Malleable cast iron	20,000	50,000	20,000	(d)	10,000	(c)	23,000,000	10,500,000	10.0	80
Ingot iron, annealed (Commercial pure iron)	16,000	42,000	19,000	(d)	12,000	30,000	30,000,000	12,000,000	48.0	69
Commercial Wrought iron	30,000	50,000	30,000	(d)	18,000	35,000	27,000,000	10,000,000	35.0	100
Steel, 0.10% carbon, hot-rolled	30,000	50,000	30,000	(d)	18,000	35,000	30,000,000	12,000,000	45.0	100
0.20% carbon, hot-rolled	35,000	60,000	35,000	(d)	21,000	45,000	30,000,000	12,000,000	35.0	120
cold-rolled or cold-drawn	60,000	80,000	60,000	(d)	36,000	60,000	30,000,000	12,000,000	18.0	160
annealed castings	30,000	60,000	30,000	(d)	18,000	45,000	30,000,000	12,000,000	30.0	120
Steel, 0.40% carbon, hot-rolled	40,000	70,000	40,000	(d)	24,000	55,000	30,000,000	12,000,000	25.0	135
heat-treated for fine grain	60,000	90,000	60,000	(d)	36,000	75,000	30,000,000	12,000,000	25.0	210
annealed castings	32,000	65,000	30,000	(d)	18,000	55,000	30,000,000	12,000,000	15.0	130
Steel, 0.60% carbon, hot-rolled	60,000	100,000	60,000	(d)	36,000	80,000	30,000,000	12,000,000	15.0	200
heat-treated for fine grain	75,000	120,000	75,000	(d)	45,000	100,000	30,000,000	12,000,000	15.0	235
Steel, 0.80% carbon, hot-rolled	70,000	120,000	70,000	(d)	42,000	105,000	30,000,000	12,000,000	10.0	240
heat-treated for fine grain	90,000	150,000	90,000	(d)	54,000	125,000	30,000,000	12,000,000	9.0	300
oil-quenched	120,000	180,000	120,000	(d)	72,000	150,000	30,000,000	12,000,000	2.0	370
Steel, 1.00% carbon, hot-rolled	80,000	135,000	80,000	(d)	48,000	115,000	30,000,000	12,000,000	10.0	260
heat-treated for fine grain	120,000	175,000	120,000	(d)	72,000	148,000	30,000,000	12,000,000	9.0	350
oil-quenched	135,000	220,000	135,000	(d)	80,000	185,000	30,000,000	12,000,000	1.0	430
Structural Nickel steel, 3.5% nickel, hot-rolled	55,000	95,000	55,000	(d)	33,000	80,000	30,000,000	12,000,000	25.0	190
Nickel steel, 3.5% nickel, 0.40% carbon, maximum hardness for machineability	150,000	170,000	150,000	(d)	90,000	140,000	30,000,000	12,000,000	9.0	350
Oil-quenched	160,000	285,000	160,000	(c)	96,000	(c)	30,000,000	12,000,000	5.0	500
Other alloy steels for machine parts, see special note below.										
Silico-manganese steel, 1.95% Si., 0.70% Mn Spring tempered	130,000	174,000	130,000	(c)	78,000	115,000	30,000,000	12,000,000	14.0	360
Nickel Steel, 15% nickel	140,000	170,000	(c)	(c)	(c)	(c)	(c)	(c)	Slight	(c)
Wire (not annealed) (e):										
Iron or soft steel	70,000	85,000	(f)	(f)	(f)	(f)	30,000,000	(f)	(c)	(c)
High-carbon steel	150,000	200,000	(f)	(f)	(f)	(f)	30,000,000	(f)	(c)	(c)

Special Note on Alloy Steels: The strength properties of the various alloy steels used for machine parts—such as chrome-nickel steel, chrome-vanadium steel, molybdenum steel, etc.—depend on the heat-treatment used rather than on the special ingredients present. The range of strength values for any of the above-named steels is about the same as for nickel steel.

(a) No well-defined proportional elastic limit. (b) Greater than the tensile strength: under pure shear cast iron falls by tension on an oblique plane.
(c) Data lacking. (d) For ductile material the yield-point, which is slightly greater than the proportional elastic limit, is the practical ultimate in compression.
(e) For annealed wire the values are about the same as for the corresponding grade of rolled steel. (f) Wire can offer resistance only in tension.

TABLE II

Average Values for Static Strength and Ductility of Various Non-ferrous Metals and Alloys

Values given are based on test data from various testing laboratories.

Metal or Alloy	Approximate Composition Per Cent	Weight Lb. per Cu. In.	Strength in Tension (a)		Elongation in 2 In. Per Cent
			Proportional Elastic Limit Lb. per Sq. In.	Ultimate Tensile Strength Lb. per Sq. In.	
Copper, annealed	Copper, 100	0.320	3,200	32,000	56
cold-drawn			38,000	50,000	8
Zinc, cast	Zinc, 100	0.253	....	9,000	Slight
rolled			5,000	24,000	50
Tin, cast	Tin, 100	0.260	....	4,000 (b)	....
rolled			....	5,000 (c)	....
Lead, cast	Lead, 100	0.410	....	1,700 (d)	....
rolled			....	3,300 (e)	....
Common brass, cast	Copper, 60; zinc, 40	0.290	20,000	55,000	20
rolled			25,000	65,000	30
Phosphor bronze,					
cast	Copper, 95; tin, 4.9; phosphorus trace	0.32	10,000	32,000	7
rolled			40,000	65,000	30
hard-drawn			....	105,000	5
spring wire					
Aluminum bronze,					
cast	Copper, 90; aluminum, 10	0.27	25,000	60,000	25
rolled			30,000	70,000	30
Monel metal,	Nickel, 67: copper, 28; iron + carbon + manganese + silicon, 5	0.32			
cast			37,000	72,000 (f)	34
rolled			50,000	85,000	42

General Note: A large number of special alloys of copper, tin, zinc, and other metals are in use whose physical properties differ somewhat from those tabulated here. The physical properties tabulated above are only general averages, and they may be materially modified by heat treatment, and very markedly modified by cold-rolling or cold drawing.

(a) Where no values for compressive strength are noted the ultimate in compression may be assumed as having the same value as the proportional limit in tension. The strength in shear may be taken as 60 per cent of the strength in tension.

(b) Ultimate in compression, 6400 lb. per sq. in.

(c) Modulus of elasticity in tension, 4,000,000 lb. per sq. in.

(d) Modulus of elasticity in tension, 700,000 lb. per sq. in.

(e) Modulus of elasticity in tension, 1,000,000 lb. per sq. in.

(f) Modulus of elasticity in tension, 22,000,000 lb. per sq. in.

TABLE III

Average Values for Static Strength and Ductility of Light Metal Alloys

The values given in this table are based on data from various testing laboratories, especially from the laboratory of the U. S. Air Service at McCook Field, Dayton, Ohio.

Metal	Weight Lb. per Cu. In.	Tensile strength* Lb. per Sq. In.		Elongation In 2 In. Per Cent
		Proportional Elastic Limit	Ultimate	
Commercial aluminum, 99% pure:				
cast	0.093	9,000	13,000	20
rolled and annealed	0.097	8,500	13,500	23
hard-drawn	0.097	20,000	30,000	4
hard-drawn wire	0.097	30,000	40,000	4
Aluminum 96%, copper 4%:				
cast	0.104	11,500	19,500	12
hard-drawn	0.104	35,000	41,000	5
Duralumin; aluminum 96%, magnesium 1%, copper 2.9%, traces of iron, silicon, and manganese:				
annealed	0.102	6,800	25,200	18
tempered	0.102	18,500	51,200	29
Electron metal, magnesium 95%, zinc 4.4%, small quantities of copper, iron, silicon, and aluminum:				
rolled	0.065	6,800	25,200	25

* The compressive strength of metals here tabulated may be safely taken as equal to the proportional elastic limit in tension. The strength in shear may safely be taken as 60 per cent of the tensile strength.

TABLE IV

General Properties and Uses of Wood

Wood	General Properties	Used for
Hard Woods:		
White Oak	Strong and tough, close grained, splits with difficulty, heavy, must be carefully seasoned to avoid checking.	Ties, vehicle and furniture making, interior finish, framing where great strength is of first consequence.
Red Oak	A softer, weaker, and more porous kind of oak than white oak.	Ties, interior finish, furniture.
Hickory	The strongest, toughest, and heaviest of American woods. Susceptible to the attacks of boring insects.	Vehicle and agricultural implement manufacture.
Maple	Rather coarse grained, but heavy and strong. It takes a fine polish.	Flooring, furniture, interior finish.
Elm	Strong and tough, but difficult to split and shape. It warps badly.	Vehicle and ship building.
Ash	Strong, but not very tough.	Interior finish.
Soft Woods:		
Spruce	Light, soft, straight grained, resistant to decay.	Framing timbers, piles, under water construction.
Douglas Fir (Oregon Fir)	Strong, though rather variable in quality, durable.	All kinds of construction.
White Pine	Light, soft, straight grained, not very strong.	Pattern making, interior finish.
Norway Pine (Red Pine)	Hard, light, coarse grained.	All kinds of construction.
Western Pine	Trade name for a number of kinds of wood with general characteristics somewhat like Norway pine.	General construction work.
Yellow Pine	The strongest and toughest of the soft woods. Heavy, decays in contact with soil.	Heavy framing timbers, flooring,
Hemlock	Light, brittle, splits easily.	Cheap framing lumber, crates and boxes.
Tamarack (Larch)	Strong, heavy, durable	Ties, sills, posts and poles, ship lumber.
Cedar	Light, durable, but not strong.	Water tanks, shingles, posts, fencing, boats.
Redwood	Durable in contact with soil. Weak and brittle, but very easy to work.	Ties, posts and poles, general construction.
Cypress	Very durable, light, close grained, easily worked, takes high polish	Shingles, poles, siding, interior finish, general building lumber.

TABLE V

Static Strength and Stiffness of Wood

The values given in this table are average values based on results of a large number of tests mainly by the U. S. Forest Service laboratories.

Wood	Weight (Dry) Lb. per Cu. Ft.	Strength and Stiffness (All Values in Lb. per Sq. In.)							
		Compression Parallel to Grain			Compression Across Grain	Shear Parallel to Grain	Flexure		
		Modulus of Elasticity	Proportional Elastic Limit	Ultimate	Proportional Elastic Limit	Ultimate	Modulus of Elasticity	Proportional Elastic Limit	Modulus of Rupture
Hardwood, small straight-grained test specimens, air dry (*a*)									
White Oak	50	....	5,000	8,500	2,200	1,000	2,000,000	9,500	13,000
Red Oak	45	....	4,500	7,000	2,000	1,000	1,950,000	9,000	11,500
Hickory	50	....	5,500	10,000	2,500	1,200	2,400,000	11,500	16,000
Maple	43	....	5,000	8,000	1,300	500	1,300,000	5,000	7,000
Soft wood, small straight-grained test specimens, air dry (*a*)									
Yellow Pine	34	1,450,000	4,000	5,800	900	800	1,650,000	6,500	11,000
Douglas Fir	28	1,100,000	3,500	5,000	800	500	1,700,000	6,500	10,500
Norway Pine and White Pine (*b*)	24	1,350,000	3,500	5,000	700	400	1,400,000	6,300	8,000
Hemlock	28.	1,900,000	4,500	5,000	600	400	1,650,000	6,300	10,000
Tamarack	31	1,350,000	3,500	4,500	700	400	1,600,000	7,500	13,000
Redwood	22	950,000	3,800	5,000	550	350	1,150,000	4,500	7,500
Spruce	24	1,000,000	5,500	7,000	700	800	1,600,000	8,000	10,000
Cedar	23	900,000	4,000	5,000	700	400	900,000	5,500	6,500
Cypress	29	....	....	6,000	800	500	1,300,000	6,500	7,500
Soft wood, full-size structural timber with ordinary defects, green.									
Yellow Pine	34	1,100,000	2,800	3,500	440	350	1,350,000	3,800	4,500
Douglas Fir	28	1,150,000	2,500	4,500	430	300	1,550,000	3,900	4,300
Norway Pine White Pine (*c*)	24	1,350,000	2,000	2,800	350	200	1,400,000	2,800	4,000
Hemlock	28	2,000,000	3,000	4,000	300	200	1,700,000	2,800	4,300
Tamarack	31	1,300,000	2,200	3,000	300	200	1,300,000	2,500	3,500
Redwood	22	950,000	2,000	2,900	350	150	900,000	2,300	2,500
Spruce	24	1,000,000	2,300	3,000	400	200	1,500,000	2,500	3,500
Cedar	23	900,000	2,000	2,600	380	130	850,000	2,800	3,400

(*a*) For most small pieces of wood it is possible to use air-dry wood and to keep it in that condition. For cases where there is likelihood of reabsorption of water the values given will be diminished by about one-third. For the rare cases in which kiln-dry wood can be used and protected from reabsorbing water the values given will be increased by about one-third.

(*b*) The wood marketed as Western pine includes several kinds of wood. It has about the same average strength values as Norway pine.

TABLE VI

Compressive Strength of Piers

The values in this table are based on test data from Watertown Arsenal, the U. S. Bureau of Standards, Cornell University, and the University of Illinois.

Make-up of Pier	Ultimate Compressive Strength Lb. per Sq. In.
Terra-cotta block, Portland cement mortar	3000
Vitrified brick, Portland cement mortar	2800
Pressed brick, Portland cement mortar	2000
Pressed brick, lime mortar	1400
Common brick, Portland cement mortar	1000
Common brick, lime mortar	700
Sand-lime brick, lime mortar	500*

* Strength of sand-lime brick piers is estimated from comparative strength of individual sand-lime bricks and common bricks.

TABLE VII

Compressive Strength of Portland Cement Concrete

The values in this table are from the report of the Joint Committee on Concrete and Reinforced Concrete. The values are based on test data from specimens in the form of cylinders 8 inches diameter by 16 inches long, stored under laboratory conditions and tested when 28 days old. The fine aggregate except for gravel and cinders is sand, and enough was used to fill the voids in the coarse aggregate; usually the ratio (by volume) of fine aggregate to coarse aggregate was about 1:2.

Coarse Aggregate	Proportion of Cement to Aggregate				
	1 : 3	1 : 4.5	1 : 6	1 : 7.5	1 : 9
All values in lb. per sq. in.					
Granite, trap rock	3300	2800	2200	1800	1400
Gravel, hard limestone, hard sandstone	3000	2500	2200	1800	1400
Soft limestone, soft sandstone	2200	1800	1500	1200	1000
Cinders	800	700	600	500	400

TABLE VIII

Results of Shearing Tests of Portland Cement Concrete

The values in this table are summaries of results given in Bulletin No. 8 of the University of Illinois Engineering Experiment Station. It is probable that concrete, a brittle material, fails in tension on inclined planes rather than in shear. The values here given are the unit-stresses in shear at failure for specimens so tested that flexure, direct tension, and direct compression were minimized. The concrete was 60 days old when tested. The aggregate used was bank sand and soft limestone.

Proportion of Cement to Aggregate	Ultimate, Lb. per Sq. In.	
	Computed Unit-stress in Shear at failure; Shearing Tests	Compressive Strength as Given by Compression Tests
1 : 6	1290	2430
1 : 9	1090	1290

TABLE IX

Static Strength and Stiffness of American Building Stone

Values based on test data from Watertown Arsenal.

Stone	Weight (Av.) Lb. per Cu. Ft.	Ultimate in Compression Lb. per Sq. In.			Computed Ultimate in Shear Lb. per Sq. In. Av.	Modulus of Rupture (Flexure) Lb. per Sq. In.			Modulus of Elasticity (Flexure) Lb. per Sq. In.
		Max.	Min.	Av.		Max.	Min.	Av.	
Granite....	165	26,000	15,000	20,000	2300	2200	1200	1600	7,500,000
Marble.....	170	16,100	10,300	12,500	1300	2300	800	1500	8,200,000
Limestone..	160	20,000	3,200	9,000	1400	2700	250	1200	8,400,000
Sandstone..	135	19,000	6,700	12,500	1700	2200	500	1500	3,300,000
Slate.......	175	......	......	15,000	....	......	......	8500	14,000,000

TABLE X

Working Stresses for Structural Materials Subjected to Static Loading

The values in this table are from the building ordinances of the city of Chicago.

Allowable Compressive Stresses for Masonry (lb. per sq. in.):

Coursed rubble, Portland cement mortar	200
Ordinary rubble, Portland cement mortar	100
Coursed rubble, lime mortar	120
Ordinary rubble, lime mortar	60
First-class granite masonry, Portland cement mortar	600
First-class limestone and sandstone masonry, Portland cement mortar	400
Portland cement concrete, 1–2–4 mixture, machine mixed	400
Portland cement concrete, 1–2½–5 mixture, machine mixed	350
Portland cement concrete, 1–3–6 mixture, machine mixed	300
Paving brick masonry, Portland cement mortar	350
Selected hard common brick masonry, Portland cement mortar	200
Common brick masonry, Portland cement mortar	175
Common brick masonry, lime mortar	100

Allowable Stresses for Timber (lb. per sq. in.):

Wood	Extreme Fibers in Beams	Compression Along the Grain*	Compression Across the Grain	Shear Along the Grain
Douglas fir and long leaf yellow pine	1300	1100	250	130
Oak	1200	900	500	200
Short leaf yellow pine	1000	800	250	120
Norway pine and white pine	800	700	200	80
Hemlock	600	500	150	60

* For columns these values are too high; a column formula must be used.

Allowable Stresses for Iron and Steel (lb. per sq. in.):

Kind of Stress	Rolled Steel	Steel Castings	Wrought Iron	Cast Iron
Tension on net section	16,000	16,000	12,000	....
Max. compression on gross section	14,000	14,000	10,000	10,000
Bending on extreme fiber	16,000	16,000	12,000	....
Bending on extreme fiber, tension	....	....	....	3,000
Bending on extreme fiber, compression	....	....	....	10,000
Bending on extreme fiber, pins	25,000	....	....	....
Shear, pins and shop-driven rivets	12,000	....	....	....
Shear, field-driven rivets	10,000	....	....	....
Shear on rolled steel shapes	12,000	....	....	....
Shear plate girder webs, gross section	10,000	....	....	....
Shear on brackets	....	....	....	2,000
Bearing, pins and shop-driven rivets	25,000	....	....	....
Bearing, field-driven rivets	20,000	....	....	....

TABLE XI

Average Values of Modulus of Resilience (or Elastic Energy Strength) and Toughness (or Ultimate Energy Strength)

Material	Tensile Proportional Limit (lb. per sq. in.) s_e	Tensile Ultimate Strength (lb. per sq. in.) s_u	Tensile Modulus of Elasticity (lb. per sq. in.) E	Ultimate Elongation Per Inch of Length (in.) ϵ_u	Tensile Modulus of Resilience (in.-lb. per cu. in.) $\frac{1}{2}\frac{s_e^2}{E}$	Toughness in Tension (Represented by area under Stress-strain Diagram) (in.-lb. per cu. in.)
Low carbon steel	30,000	60,000	30,000,000	0.35	15.0	15,700
Medium carbon steel	45,000	85,000	30,000,000	0.25	33.7	16,300
High carbon steel	75,000	120,000	30,000,000	0.08	94.0	5,100
Special alloy steel (Heat treated)	200,000	230,000	30,000,000	0.12	667.0	22,000
Gray cast iron	6,000	20,000	15,000,000	0.005	1.2	70
Malleable cast iron	20,000	50,000	23,000,000	0.10	17.4	3,800
Rolled bronze	40,000	65,000	14,000,000	0.30	57.2	15,500
Timber (Hickory)	5,500*	10,000*	2,400,000*	......	6.32*	

* In compression.

TABLE XII

Values of Endurance Limit, Ultimate Strength, Proportional Limit, and of the Ratio of the Endurance Limit to the Ultimate Strength and to the Proportional Limit for Various Grades of Steel

(Specimens were subjected to completely reversed cycles of bending stress and we-e free from abrupt changes of section or surface scratches.)

Description of Steel		Endurance Limit, Lb. per Sq. In., s_r	Ultimate Tensile Strength, Lb. per Sq. In., s_u	Tensile Proportional Limit, Lb. per Sq. In., s_e	Ratio	
Composition of steel	Treatment				$\frac{s_r}{s_u}$	$\frac{s_r}{s_e}$
0.02% C (Ingot Iron)	As received (hot rolled)	25,000	42,400	16,100	0.58	1.56
0.21% C	As received (hot rolled)	34,000	70,000	39,900	.48	0.85
0.20% C	Cold drawn (surface polished after drawing)	41,000	80,800	55,200	.47	.72
0.30% C	Annealed*	30,000	69,900	35,000	.43	.86
0.57% C	Normalized*	33,000	71,900	34,500	.46	.96
0.37% C	Hardened† and Tempered	45,000	94,200	61,500	.48	.73
0.49% C	Hardened† and Tempered	48,000	96,900	67,700	.50	.71
0.77% C	Annealed	39,000	111,100	46,300	.35	.88
0.93% C (Spg. Steel)	Annealed‡	30,500	84,100	28,000	.36	1.09
0.93% C " "	Hardened and Tempered	56,000	97,100	84,800	.58	0.67
0.93% C " "	Hardened	98,000	188,300	106,500	.52	.92
1.20% C	Normalized‡	50,000	116,900	55,300	.43	.90
1.20% C	Hardened and Tempered	92,000	179,900	100,700	.51	.91
3.5% Nickel	Hardened and Tempered	63,000	118,000	86,400	.56	.73
3.5% Nickel	Annealed	54,000	101,600	60,800	.53	.89
Chrome-Nickel 3.3% Ni., 0.24% C, 0.87% Cr.	Hardened and Tempered	68,000	138,700	122,600	.49	.57
Chrome-Nickel 3.3%, Ni., 0.24% C, 0.87% Cr	Annealed	49,000	87,300	56,700	.56	.86
Nickel-Molybdenum	Hardened and Tempered	57,000	133,300	100,000	.43	.57
Chrome-Molybdenum	Hardened and Tempered	67,000	141,400	112,500	.48	.60
Chrome-Vanadium	Hardened and Tempered	67,000	146,600	114,000	.46	.59
Selico-Manganese (Spring steel)	Softer than is usual for springs	62,000	157,400	100,000	.40	.62

* Annealing consists in heating the steel above the upper critical temperature (about 800° C.) and then cooling the steel very slowly, as for example, cooling in the furnace after the fire is drawn. Normalizing differs from annealing in that the rate of cooling is somewhat faster, the material is usually drawn from the furnace and allowed to cool in the air.

† Hardening consists in heating the steel above its upper critical temperature and cooling it rapidly by quenching it in water or oil. Tempering consists in heating the material, after it has been quenched, to 450° C., more or less, to reduce the hardness slightly and to give the material a more uniform internal structure.

‡ This steel would seldom be used except in the hardened and tempered condition; it is included in the table mainly to show the effect of heat treatment.

APPENDIX I

FIRST MOMENTS AND CENTROIDS OF AREAS

158. Definitions.—The *moment of an area* with respect to an axis is the algebraic sum of the moments of the elementary parts of the area, the moment of each part being the product of the elementary area and the perpendicular distance from the elementary area to the axis. This moment is called the *first* moment to distinguish it from the moment of inertia (or second moment, see Appendix II) of the area. The first moment of an area is sometimes called the statical moment of the area.

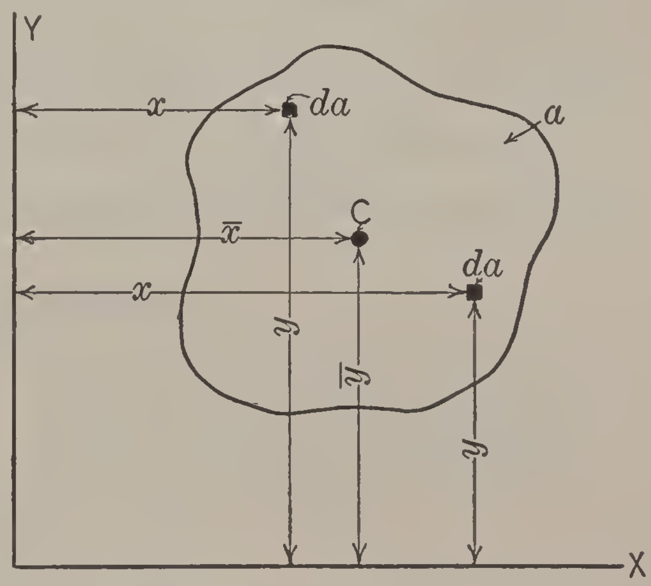

Fig. 274.

The *centroid* of an area is a point whose distance (called a centroidal distance) from any axis times the total area is equal to the moment of the area with respect to that axis. Hence, the coordinates, $\bar{x}$ and $\bar{y}$, of the centroid C of an area a (Fig. 274) may be found from the equations

$$a\bar{x}=\int xda \quad \text{and} \quad a\bar{y}=\int yda. \quad . \quad . \quad . \quad . \quad (242)$$

Or, expressed in another way, the centroid of an area is that point at which the whole area may be conceived to be concentrated and have the same moment with respect to any axis as has the actual (distributed) area.

If an area is symmetrical with respect to an axis, the centroid of the area lies in the given axis. This statement is evident from the fact that the moments of the areas on the opposite sides of the axis are numerically equal but of opposite sign. If an area is symmetrical with respect to each of two axes, the centroid of the area is the point of intersection of the two axes.

159. Centroids Found by Integration.—In determining the centroid of an area by the method of integration, from the equations in the preceding article, it is possible to select the element of area in various ways and to express the element in terms of either Cartesian or polar coordinates. The resulting integral may be a single or a double integral, depending on the way the element is selected. The integral, of course, is a definite integral, the limits of integration depending on the boundary curve of the area. In any case the element of area must be taken so that:

1. All points of the element are the same distance from the line about which moments are taken; otherwise, the distance from the line to the element will be indefinite.

2. The centroid of the element is known, in which case the moment of the element about the moment axis is the product of the element and the distance of its centroid from the axis or plane.

The centroids of some of the common areas will be found in the following illustrative problems:

ILLUSTRATIVE PROBLEMS

Find, by the method of integration, the centroids of the following areas with respect to the axes indicated.

Problem 246. Area of a Triangle.—In accordance with the second of the above rules the elements of area will be taken as strips parallel to the base of the triangle (Fig. 275). Since each element is bisected by the medium drawn from the vertex opposite the base, the centroid of each element, and hence of the entire area, lies on this median. If x denotes the width of the strip, the area of the strip is $da = xdy$. Thus.

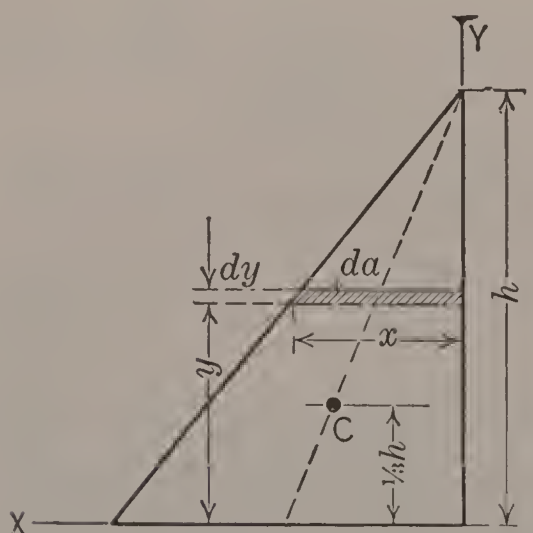

FIG. 275.

$$a\bar{y} = \int xydy.$$

From similar triangles, the relation between x and y is,

$$\frac{x}{h-y} = \frac{b}{h} \quad \text{or} \quad x = \frac{b}{h}(h-y).$$

Hence,

$$a\bar{y} = \frac{b}{h}\int_0^h (h-y)y\,dy = \tfrac{1}{6}bh^2.$$

Therefore,

$$\bar{y} = \frac{\tfrac{1}{6}bh^2}{\tfrac{1}{2}bh} = \frac{1}{3}h.$$

The centroid of a triangular area, then, is on the median line at a distance of one-third of the altitude from the base.

Problem 247. Sector of a Circle.—*First Method.*—The element of area will be selected in accordance with the first of the above rules as indicated in Fig. 276. Since the area is symmetrical with respect to the x-axis, the centroid lies on this axis, and hence $\bar{y}=0$. The value of $\bar{x}$ may then be found from the equation,

$$a\bar{x} = \int x\,da$$

$$= \int_0^r \int_{-\alpha}^{+\alpha} \rho \cos\theta \cdot \rho\,d\rho\,d\theta = \tfrac{2}{3}r^3 \sin\alpha.$$

Therefore,

$$\bar{x} = \frac{\tfrac{2}{3}r^3 \sin\alpha}{a} = \frac{\tfrac{2}{3}r^3 \sin\alpha}{r^2\alpha} = \frac{2}{3}\,\frac{r \sin\alpha}{\alpha}.$$

If $\alpha = 90° = \frac{\pi}{2}$ radians, that is, if the sector is a semicircular area,

$$\bar{x} = \frac{4r}{3\pi}.$$

as indicated in Fig. 277.

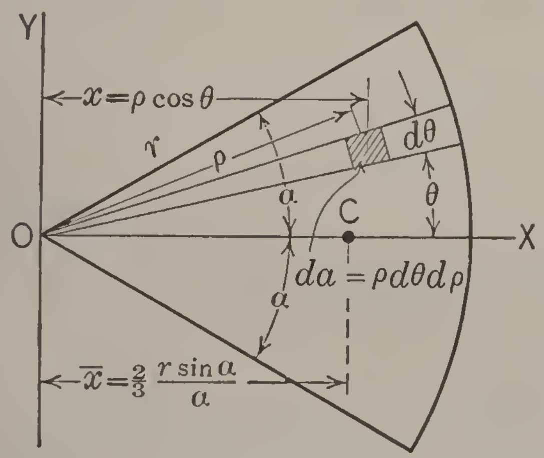

Fig. 276.

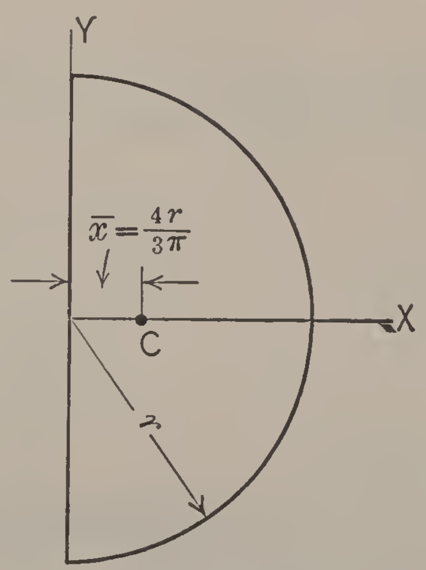

Fig. 277.

Second Method.—In accordance with the second of the above rules, the element of area will be selected as a triangle, as indicated in Fig. 278. The area of the triangle is $\frac{1}{2}r^2 d\theta$ and the distance of its centroid from the y-axis is

$\frac{2}{3}r \cos \theta$. Hence, the moment of the triangle with respect to the y-axis is $\frac{1}{3}r^3 \cos \theta d\theta$, and $\bar{x}$ is obtained from the equation,

$$a\bar{x} = \int_{-\alpha}^{+\alpha} \tfrac{1}{3}r^3 \cos \theta d\theta$$

$$= \tfrac{2}{3}r^3 \sin \alpha.$$

Therefore,

$$\bar{x} = \frac{\frac{2}{3}r^3 \sin \alpha}{r^2\alpha} = \frac{2}{3}\,\frac{r \sin \alpha}{\alpha}.$$

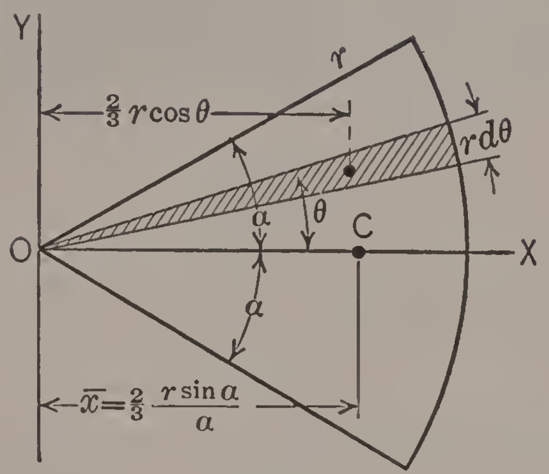

FIG. 278.

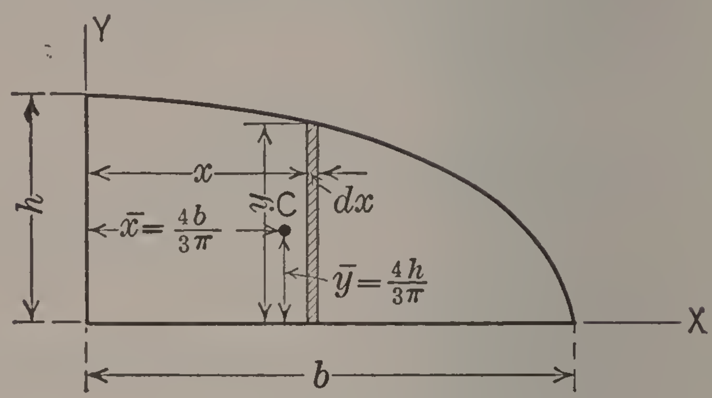

FIG. 279.

Problem 248. Area of Quadrant of an Ellipse.—The semi-axes of the ellipse will be denoted by b and h (Fig. 279) and hence the equation of the ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{h^2} = 1.$$

A strip parallel to the y-axis will be selected as the element of area. From the equation of the ellipse, y may be expressed in terms of x by the equation,

$$y = \frac{h}{b}\sqrt{b^2 - x^2}.$$

Hence,

$$a = \int_0^b y dx = \frac{h}{b}\int_0^b \sqrt{b^2 - x^2} dx = \tfrac{1}{4}\pi bh.$$

To find $\bar{x}$;

$$\tfrac{1}{4}\pi bh \cdot \bar{x} = \int x da$$

$$= \int_0^b x \cdot y dx = \frac{h}{b}\int_0^a x\sqrt{b^2 - x^2} dx = \tfrac{1}{3}b^2 h.$$

Therefore,

$$\bar{x} = \frac{\frac{1}{3}b^2 h}{\frac{1}{4}\pi bh} = \frac{4b}{3\pi}.$$

To find $\bar{y}$; the same strip will be used for the element of area, its centroid being at the distance $\frac{1}{2}y$ from the x-axis. Thus,

$$\tfrac{1}{4}\pi bh \cdot \bar{y} = \int_0^a \tfrac{1}{2}y \cdot y dx = \frac{h^2}{2b^2}\int_0^b (b^2 - x^2)dx = \tfrac{1}{3}bh^2.$$

Therefore,

$$\bar{y} = \frac{\frac{1}{3}bh^2}{\frac{1}{4}\pi bh} = \frac{4h}{3\pi}.$$

Problem 249—Parabolic Segment.—Let the segment be bounded by the x-axis, the line $x = a$, and the parabola $y^2 = \dfrac{h^2 x}{b}$ as shown in Fig. 280. A

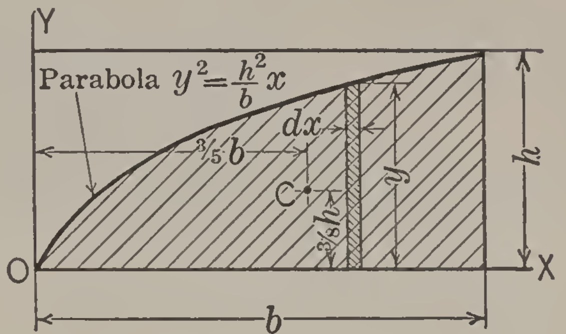

Fig. 280.

strip parallel to the y-axis will be selected as the element of area, the area of the strip being expressed by ydx. The area of the segment, then, is

$$a = \int da = \int_0^b ydx = \frac{h}{\sqrt{b}}\int_0^b \sqrt{x}dx$$

$$= \frac{h}{\sqrt{b}}\left[\frac{2}{3}x^{3/2}\right]_0^b = \frac{2}{3}bh.$$

To find $\bar{x}$;

$$a\bar{x} = \int xda$$

$$= \int_0^b xydx = \frac{h}{\sqrt{b}}\int_0^b x^{3/2}dx = \frac{2}{5}hb^2.$$

Therefore,

$$\bar{x} = \frac{\frac{2}{5}hb^2}{a} = \frac{\frac{2}{5}hb^2}{\frac{2}{3}bh} = \frac{3}{5}b.$$

To find $\bar{y}$; the same elementary strip will be selected, but since each point of the element is not the same distance from the x-axis, its moment must be expressed as the product of the area of the strip and its centroidal distance, $\dfrac{y}{2}$, from the x-axis. Thus,

$$a\bar{y} = \int_0^b \frac{1}{2}y \cdot ydx = \frac{1}{2}\frac{h}{b}\int_0^b xdx = \frac{1}{4}h^2 b.$$

Therefore,

$$\bar{y} = \frac{\frac{1}{4}h^2b}{a} = \frac{\frac{1}{4}h^2b}{\frac{2}{3}bh} = \frac{3}{8}h.$$

Problem 250. Shaded Area in Fig. 281.—Let it be required to locate the centroid of the area between the y-axis, the line $y=h$, and the parabola $y^2 = \frac{h^2x}{b}$ represented by the shaded area in Fig. 281.

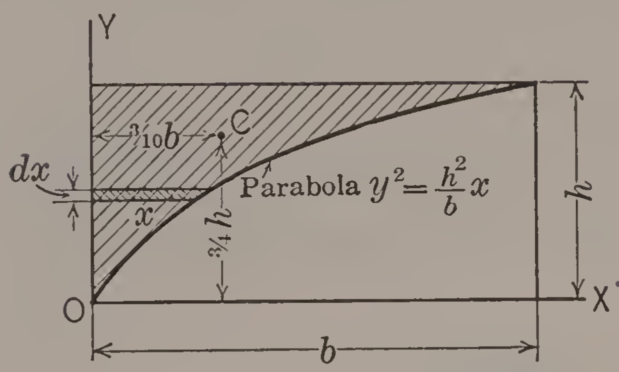

FIG. 281.

The element of area shown in the figure is parallel to the x-axis, and is equal to xdy. The shaded area a is the area (bh) of the rectangle minus the area ($\frac{2}{3}bh$) of the parabola segment. Hence $a = \frac{1}{3}bh$.

To find $\bar{x}$: In accordance with the second of the rules in Art. 159, we have

$$a\bar{x} = \int_0^h \left(xdy \cdot \frac{x}{2}\right) = \frac{1}{2}\int_0^h x^2dy$$

$$= \frac{1}{2}\int_0^h \frac{y^4b^2}{h^4}dy = \frac{1}{2}\frac{b^2}{h^4}\frac{h^5}{5} = \frac{1}{10}b^2h.$$

Therefore

$$\bar{x} = \frac{3}{10}b.$$

To find $\bar{y}$:

$$a\bar{y} = \int yda = \int yxdy$$

$$= \frac{b}{h^2}\int_0^h y^3dy = \frac{b}{h^2}\frac{h^4}{4} = \frac{bh^2}{4}.$$

Therefore,

$$\bar{y} = \frac{3}{4}h.$$

160. Centroids of Composite Areas.—As noted in Art. 158, if the centroid of an area is known, the moment with respect to an axis is most easily found by multiplying the area by the distance of the centroid from the axis. Thus, if a given area can be divided into parts, the centroids of which are known, the moment of the whole area may be found, without integrating, by obtaining the algebraic sum of the moments of the parts into which the area is divided, the moment of each part being the product of that part and the distance of its centroid from the line. Thus, for example, if a'_1, a'_2, a'_3, etc., denote the parts into which an area a is divided, and x'_0, x''_0, x'''_0, etc., denote the x-coordinates of the centroids of the respective parts, then,

$$(a'_1+a'_2+a'_3+\ \ldots\ )\bar{x}=a'_1x'_0+a'_2x''_0+a'_3x'''_0, \quad . \quad (243)$$

or,

$$a\bar{x}=\Sigma(a'x_0).$$

Similarly,

$$a\bar{y}=\Sigma(a'y_0).$$

ILLUSTRATIVE PROBLEM

Problem 251. Locate the centroid of the T-section shown in Fig. 282.

Solution.—If axes be selected as indicated, it is evident from symmetry that $\bar{x}=0$. By dividing the given area into areas a'_1 and a'_2, and by taking moments about the bottom edge of the area, $\bar{y}$ may be found as follows:

$$a\bar{y}=\Sigma(a'y_0),$$

$$\bar{y}=\frac{12\times1+12\times5}{6\times2+6\times2}=3 \text{ in.}$$

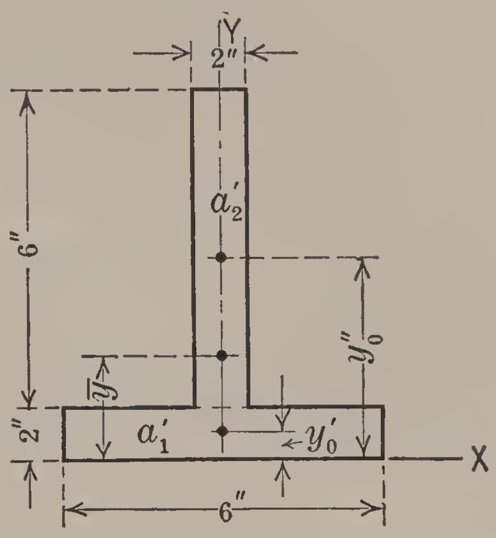

FIG. 282.

PROBLEMS

252. Locate the centroid of the channel section shown in Fig. 283.

Ans. $\bar{x}=0.79$ in.

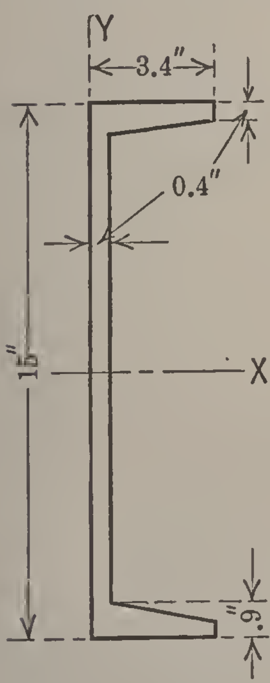

FIG. 283.

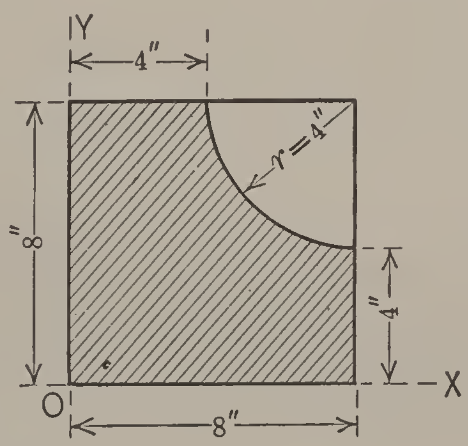

FIG. 284.

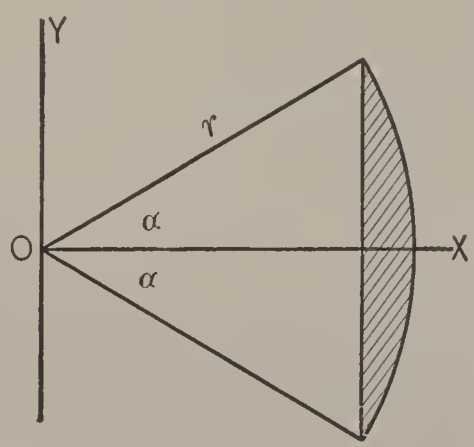

FIG. 285.

253. Locate the centroid of the shaded area shown in Fig. 284.

254. Locate the centroid of the segment of a circle as shown in Fig. 285.

In the expression for $\bar{x}$ make $\alpha=\frac{\pi}{2}$ and see if the result agrees with the result found in Prob. 247 for a semicircle.

255. Find the distance, from the larger base, of the centroid of the area of the trapezoid shown in Fig. 286. *Ans.* $y=\frac{h}{3}\cdot\frac{2b_1+b_2}{b_1+b_2}$.

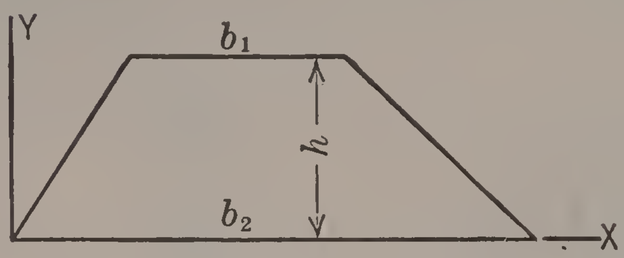

FIG. 286.

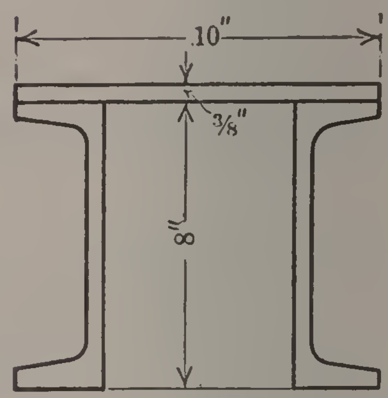

FIG. 287.

256. Fig. 287 represents the cross-section of the end post of a bridge. The area of each channel section is 4.78 square inch. Find the distance from the top of the cover plate to the centroid of the section.

APPENDIX II

SECOND MOMENT OR MOMENT OF INERTIA OF AN AREA

161. Moment of Inertia of an Area Defined.—In the analysis of many engineering problems as, for example, in determining the stresses in a beam or a shaft, expressions of the form $\int x^2 da$ are frequently met, in which da represents an element of an area a, and x is the distance of the element from some axis in, or perpendicular to, the plane of the area, the limits of integration being such that each element of the area is included in the integration. An expression of this form is called the *second moment* of the area or the *moment of inertia* of the area with respect to the given axis.

The moment of inertia of an area with respect to an axis in, or perpendicular to, the plane of the area may, then, be defined as the sum of the products obtained by multiplying each element of the area by the square of its distance from the given axis.

The term moment of inertia is somewhat misleading, since inertia is a property of physical bodies, only, and hence an area does not possess inertia. For this reason the term, second moment of an area, is to be preferred, particularly when contrasting the expressions of the form here discussed with expressions which were defined as first moments of areas in Appendix I. It may be noted that each term x^2da in the summation can be written in the form $x(xda)$, and hence represents the moment of the moment of an element of area, that is, the second moment of the element. The term moment of inertia, however, is very widely used, due to the fact that the expression is of the same form as an expression which is defined as the moment of inertia of a body and which does have a physical significance.

The moment of inertia of an area with respect to an axis will be denoted by I for an axis in the plane of the area and by J for an axis perpendicular to the plane of the area. The particular axis (or direction of the axis) about which the moment of inertia is taken will be denoted by subscripts. Thus, the moments of inertia of the area (Fig. 288) with respect to the x- and y-axes are expressed as follows:

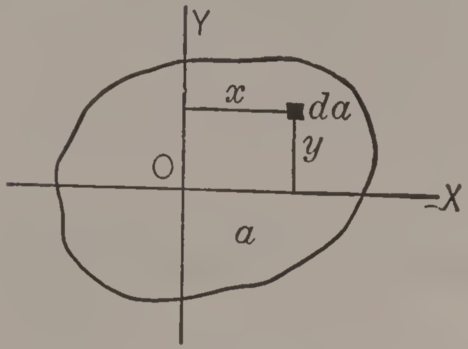

FIG. 288.

$$\left.\begin{aligned} I_x &= \int y^2 da, \\ &\text{and} \\ I_y &= \int x^2 da. \end{aligned}\right\} \qquad (244)$$

Units and Sign.—Since the moment of inertia of an area is the sum of a number of terms each of which is the product of an area and the square of a distance, the moment of inertia of an area is expressed as a length to the fourth power. If, then, the inch (or foot) be taken as the unit of length, the moment of inertia will be expressed as inches (or feet) to the fourth power (written in.^{4} or ft.^{4}). Further, the sign of each of the products x^2da is always positive since x^2 is always positive, whether x is positive or negative, and da is essentially positive. Therefore the moment of inertia, or second moment, of an area is always positive. In this respect it differs from the first moment of an area, which may be positive, negative, or zero, depending on the position of the moment axis.

162. Polar Moment of Inertia.—The moment of inertia of an area with respect to a line perpendicular to the plane of the area is called the *polar moment of inertia* of the area and, as noted in Art. 161, will be denoted by J. Thus the polar moment of inertia with respect to the z-axis, of an area in the xy-plane (Fig. 288) may be expressed as follows:

$$\begin{aligned} J_z &= \int r^2 da \\ &= \int (x^2+y^2) da \\ &= \int x^2 da + \int y^2 da. \end{aligned}$$

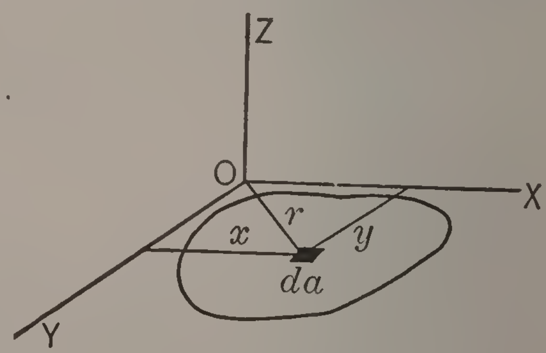

FIG. 289.

Therefore,

$$J_z = I_y + I_x. \qquad (245)$$

Hence the following proposition may be stated:

The polar moment of inertia of an area with respect to any axis is equal to the sum of the moments of inertia of the area with respect to any two rectangular axes in the plane of the area which intersect on the given polar axis.

163. Radius of Gyration.—Since the moment of inertia of an area $\left(\int x^2da \text{ or } \int r^2da, \text{ etc.,}\right)$ is four dimensions of length, it may be expressed as the product of the total area, a, and the square of a distance, k. Thus,

$$\left.\begin{aligned} I_x &= \int y^2da = ak_x^2, \\ \text{or} \qquad J_z &= \int r^2da = ak_z^2. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (246)$$

The distance k is called the radius of gyration of the area with respect to the given axis, the subscript denoting the axis with respect to which the moment of inertia is taken. The radius of gyration of an area with respect to a line, then, may be defined as a distance such that, if the area were conceived to be concentrated at this distance from the given line, the moment of inertia would be the same as the moment of inertia of the actual or distributed area with respect to the same line.

From the equation $I_y = \int x^2da = ak_y^2$, it will be noted that k_y^2, the square of the radius of gyration with respect to the y-axis, is the mean of the squares of the distances, from the y-axis, of the equal elements of area into which the given area may be divided, and that it is *not* the square of the mean of these distances. The mean distance $(\bar{x})$ of the elements of area from the y-axis is the centroidal distance as discussed in the preceding chapter. Hence $a\bar{x}^2$ does *not* represent the moment of inertia of an area with respect to the y-axis.

164. Parallel Axis Theorem.—If the moment of inertia of an area with respect to a centroidal axis in the plane of the area is known, the moment of inertia with respect to any parallel axis in the plane may be determined, without integrating, by means of a proposition which may be established as follows: In Fig. 290 let YY be any axis through the centroid, C, of an area and let $Y'Y'$ be any axis parallel to YY and at a distance d therefrom. Further, let the moment of inertia of the area with respect to the axis YY be denoted by $\bar{I}$ and the moment of inertia with respect to $Y'Y'$ by I. By definition then,

$$I = \int (x+d)^2 da$$

$$= \int x^2 da + 2d \int x da + d^2 \int da.$$

Therefore,

$$\boldsymbol{I = \bar{I} + ad^2}, \text{ since } \int x da = a\bar{x} = 0. \quad . \quad . \quad . \quad (247)$$

Hence the following proposition may be stated:

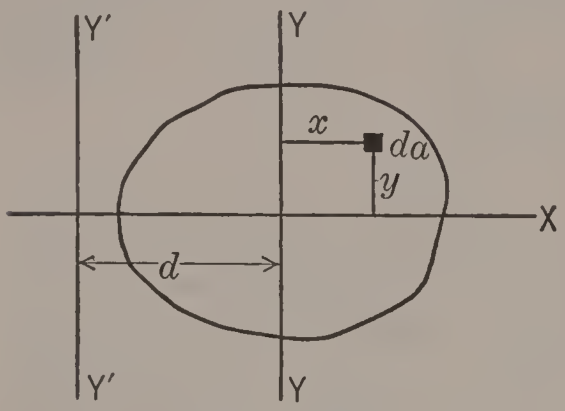

Fig. 290.

The moment of inertia of an area with respect to any axis in the plane of the area is equal to the moment of inertia of the area with respect to a parallel centroidal axis plus the product of the area and the square of the distance between the two axes. This proposition is called the *parallel axis theorem.*

A corresponding relation exists between the radii of gyration of the area with respect to two parallel axes, one of which passes through the centroid of the area. For, by replacing I by ak^2 and $\bar{I}$ by $a\bar{k}^2$ the above equation becomes,

$$ak^2 = a\bar{k}^2 + ad^2.$$

Whence,

$$\boldsymbol{k^2 = \bar{k}^2 + d^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (248)$$

where k denotes the radius of gyration of the area with respect to any axis in the plane of the area and $\bar{k}$ denotes the radius of gyration of the area with respect to a parallel centroidal axis.

Similarly, for polar moments of inertia and radii of gyration, it can be shown that,

$$J = \bar{J} + ad^2,$$

and,

$$k = \bar{k}^2 + d^2,$$

where $\bar{J}$ and $\bar{k}$ denote the polar moment of inertia and radius of gyration, respectively, of the area with respect to the centroidal axis and J and k denote the polar moment of inertia and radius of gyration, respectively, of the area with respect to an axis parallel to the centroidal axis and at a distance d therefrom.

165. Moments of Inertia Found by Integration.—In determining the moment of inertia of a plane area with respect to a line, it is possible to select the element of area in various ways and to express the area of the element in terms of either Cartesian or polar coordinates. Further, the integral may be either a single or double integral, depending on the way in which the element of area is selected; the limits of integration are determined, of course, from the boundary curve of the area. In any case, however, the elementary area must be taken so that:

(1) All points in the element are equally distant from the axis with respect to which the moment of inertia is to be found, otherwise the distance x in the expression x^2da would be indefinite. Or, so that,

(2) The moment of inertia of the element, with respect to the axis about which the moment of inertia of the whole area is to be found, is known, the moment of inertia of the area then being found by summing up the moments of inertia of the elements. Or, so that,

(3) The centroid of the element is known and also the moment of inertia of the element with respect to an axis which passes through the centroid of the element and is parallel to the given axis; the moment of inertia of the element may then be expressed by means of the parallel axis theorem.

The moments of inertia of some of the simple areas will now be found in the following illustrative problems:

ILLUSTRATIVE PROBLEMS

Problem 257.—Determine the moment of inertia of a rectangle, in terms of its base b and altitude h, with respect to (a) a centroidal axis parallel to the base; (b) an axis coinciding with the base.

Solution.—(a) **Centroidal Axis.**—The element of area will be selected in accordance with rule (1) above, as indicated in Fig. 291. The moment of inertia of the rectangular area with respect to the centroidal axis, then, is,

$$\bar{I}_x = \int y^2 da = \int_{-\frac{h}{2}}^{+\frac{h}{2}} y^2 b dy$$

$$= \frac{1}{12} bh^3.$$

(*b*) **Axis Coinciding with the Base.** *First Method.*—The element of area will be selected as indicated in Fig. 292. The moment of inertia of the rectangle with respect to the base, then, is,

$$I_b=\int y^2da=\int_0^h y^2bdy$$

$$=\frac{1}{3}bh^3.$$

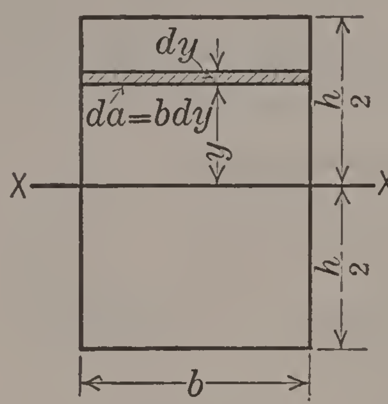

Fig. 291.

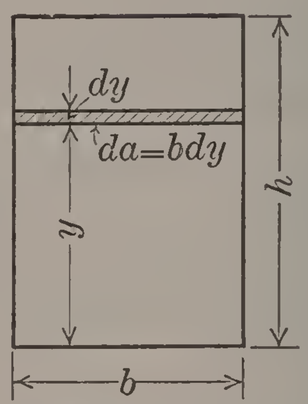

Fig. 292.

Second Method.—Since the moment of inertia of the rectangle with respect to a centroidal axis is $\frac{1}{12}bh^3$, the moment of inertia with respect to the base may be found from the parallel axis theorem (Art. 164). Thus,

$$I_b=\bar{I}_x+a\left(\frac{h}{2}\right)^2$$

$$=\tfrac{1}{12}bh^3+bh\times\frac{h^2}{4}$$

$$=\frac{1}{3}bh^3.$$

Problem 258.—Determine the moment of inertia of a triangle, in terms of its base b and altitude h, with respect to (*a*) an axis coinciding with its base; (*b*) a centroidal axis parallel to the base.

Solution.—(*a*) **Axis Coinciding with the Base.**—The elementary area will be selected as shown in Fig. 293. The moment of inertia of the area of the triangle with respect to the base, then, is,

$$I_b=\int y^2da=\int y^2xdy.$$

But, from similar triangles,

$$\frac{x}{b}=\frac{h-y}{h}.$$

Hence,

$$x=\frac{b}{h}(h-y).$$

Therefore,

$$I_b = \frac{b}{h}\int_0^h y^2(h-y)dy$$

$$= \frac{1}{12}bh^3.$$

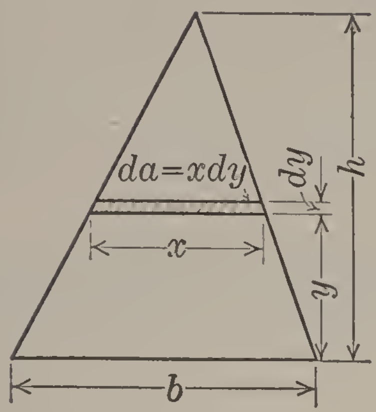

FIG. 293.

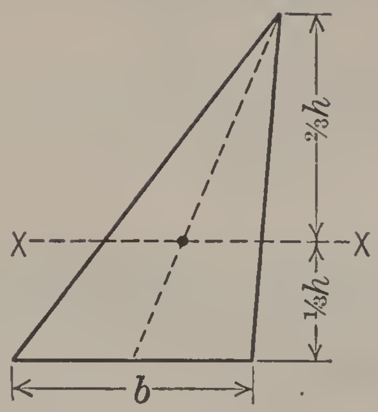

FIG. 294.

(*b*) **Centroidal Axis Parallel to the Base.**—The centroidal axis parallel to the base (axis XX) is shown in Fig. 294 (see Prob. 246). Using the parallel axis theorem, the moment of inertia of the triangular area with respect to the centroidal axis is,

$$\bar{I}_x = I_b - a(\tfrac{1}{3}h)^2$$

$$= \tfrac{1}{12}bh^3 - \tfrac{1}{2}bh \times \tfrac{1}{9}h^2$$

$$= \frac{1}{36}bh^3.$$

Problem 259. Determine the moment of inertia of the area of a circle, in terms of its radius r, with respect to an axis coinciding with the diameter; (*a*) using Cartesian coordinates; (*b*) using polar coordinates.

Solution.—(*a*) **Cartesian Coordinates.**—The element of area will be selected as shown in Fig. 295. The moment of inertia of the circular area with respect to the diameter, then, is,

$$\bar{I}_x = \int y^2 da = \int y^2 2x dy$$

$$= 2\int_{-r}^{+r} y^2\sqrt{r^2-y^2}dy$$

$$= \frac{1}{4}\pi r^4.$$

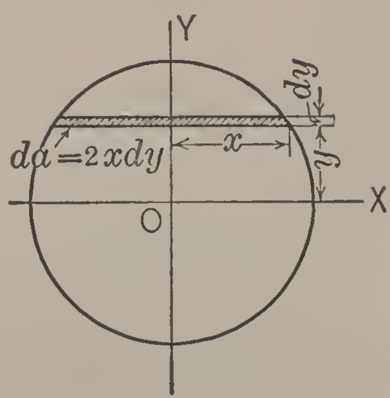

FIG. 295.

(*b*) **Polar Coordinates.**— The element of area will be selected as shown in Fig. 296. Hence,

$$\bar{I}_x = \int y^2 da,$$

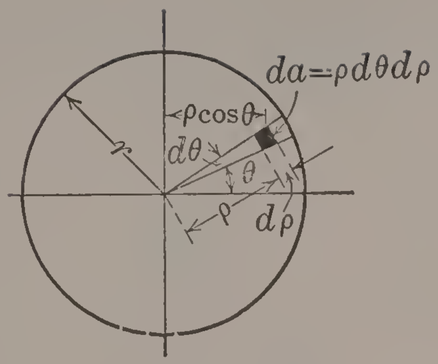

FIG. 296.

$$\bar{I}_x=\int_0^r\int_0^{2\pi}(\rho\sin\theta)^2\,\rho d\rho d\theta$$

$$=\int_0^r\int_0^{2\pi}\rho^3\sin^2\theta\,d\rho d\theta$$

$$=\frac{r^4}{4}\int_0^{2\pi}\sin\theta\,d\theta$$

$$=\frac{r^4}{4}\times\pi=\frac{1}{4}\pi r^4=\frac{1}{64}\pi d^4.$$

Problem 260.—Determine the polar moment of inertia of the area of a circle of radius r with respect to a centroidal axis: (*a*) by integration; (*b*) by use of the theorem of Art. 132.

Solution.—(*a*) **By Integration.**—By selecting the element of area as indicated in. Fig. 297, the polar moment of inertia of the circular area is,

$$J_z=\int\rho^2 da$$

$$=\int_0^r\rho^2 2\pi\rho d\rho$$

$$=\frac{1}{2}\pi r^4=\frac{1}{32}\pi d^4.$$

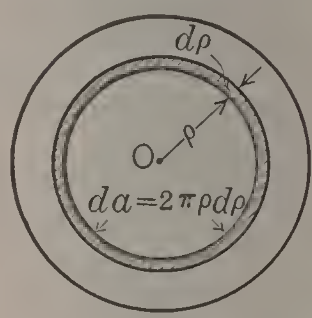

FIG. 297.

(*b*) **By Use of Theorem of Art. 162.**—Since I_x and I_y are each equal to $\frac{1}{4}\pi r^4$ (Prob. 259), the polar moment of inertia of the area of the circle is,

$$J_z=I_x+I_y$$

$$=\tfrac{1}{4}\pi r^4+\tfrac{1}{4}\pi r^4$$

$$=\frac{1}{2}\pi r^4.$$

PROBLEMS

261. Determine the moment of inertia of the area of a circle, with respec, to an axis tangent to the circle, in terms of, r, the radius of the circle.

262. Determine the polar moment of inertia of the area of a rectangle of base b and altitude h with respect to the centroidal axis.

Ans. $\bar{J}=\frac{1}{12}bh(b^2+h^2)$.

263. Find the moment of inertia and radius of gyration of a circular area 16 in. in diameter, with respect to a diameter.

264. Determine the moments of inertia of the area of an ellipse, the principal axes of which are $2b$ and $2h$, with respect to the principal axes.

Ans. $I_b=\frac{1}{4}\pi bh^3$. $I_h=\frac{1}{4}\pi hb^3$.

265. The base of a triangle is 8 in. and its altitude is 10 in. Find the moment of inertia and radius of gyration of the area of the triangle with respect to the base.

266. Find the polar moment of inertia and radius of gyration of the area of a square, each side of which is 15 in., with respect to an axis through one corner of the square.

267. Find the polar moment of inertia, with respect to a centroidal axis, of the area of an isosceles triangle having a base b and altitude h.

Ans. $J=\frac{1}{12}bh(\frac{1}{4}b^2+\frac{1}{3}h^2)$.

166. Moments of Inertia of Composite Areas.—When a composite area can be divided into a number of simple areas, such as triangles, rectangles, and circles, for which the moments of inertia are known, the moment of inertia of the entire area may be obtained by taking the sum of the moments of inertia of the several areas. Likewise, the moment of inertia of the part of an area that remains after one or more simple areas are removed may be found by subtracting, from the moment of inertia of the given area, the sum of the moments of inertia of the several parts removed.

ILLUSTRATIVE PROBLEMS

Problem 268. Locate the horizontal centroidal axis, XX, of the T-section shown in Fig. 298 and find the moment of inertia of the area with respect to this centroidal axis.

Solution. First Method.—The distance, $\bar{y}$, of the centroid of the area from the axis X_1X_1 may be found from the equation,

$$a\bar{y}=\Sigma(a'y_0).$$

Thus,

$$\bar{y}=\frac{12\times7+12\times3}{12+12}$$

$$=5 \text{ in.}$$

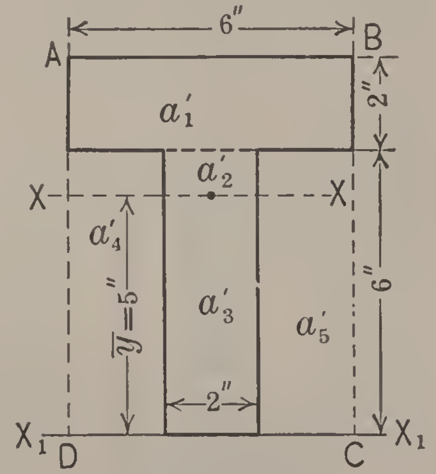

FIG. 298.

The moment of inertia with respect to the XX axis is the sum of the moments of inertia of the three parts a'_1, a'_2, and a'_3, with respect to that axis. Thus,

$$I_x=\tfrac{1}{12}\times6\times(2)^3+12\times(2)^2+\tfrac{1}{3}\times2\times(1)^3+\tfrac{1}{3}\times2\times(5)^3$$

$$=4+48+.67+83.33$$

$$=136 \text{ in.}^4$$

Second Method.—The moment of inertia of the T-section may also be determined as follows: First find the moment of inertia of the T-section with respect to the axis X_1X_1 by subtracting the moments of inertia of the parts a'_4 and a'_5 from the moment of inertia of the rectangular area $ABCD$, and then find $\bar{I}_x$ for the T-section by use of the parallel axis theorem. Thus, the moment of inertia, I_x, of the T-section with respect to the X_1X_1 axis is,

$$I_x = \tfrac{1}{3}\times 6\times(8)^3 - 2\times\tfrac{1}{3}\times 2\times(6)^3 = 736 \text{ in.}^4;$$

and

$$\bar{I}_x = I_x - Ad^2 = 736 - 24\times(5)^2 = 136 \text{ in.}^4.$$

Problem **269.** Find the moment of inertia of the channel section shown in Fig. 299 with respect to the line XX. Find also the moment of inertia with respect to the parallel centroidal axis.

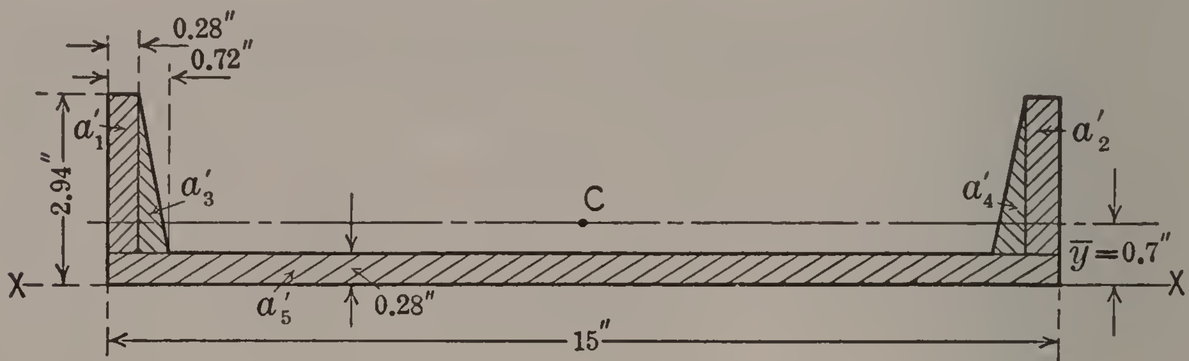

FIG. 299.

Solution.—The area may be divided into triangles and rectangles as shown in the figure. The values used in the solution may be put in tabular form as shown below, where a' denotes the area of any part, y_0 the distance of the centroid of the part from the line XX, I_0 the moment of inertia of the part with respect to its own centroidal axis parallel to XX, and I'_x the moment of inertia of the part with respect to the axis XX.

Part	a'	y_0	$a'y_0$	I_0	$a'y_0^2$	$I'_x = I_0 + ay_0^2$
a'_1	0.745	1.61	1.20	0.44	1.93	2.37
a'_2	.745	1.61	1.20	.44	1.93	2.37
a'_3	.585	1.17	0.68	.23	0.80	1.03
a'_4	.585	1.17	.68	.23	.80	1.03
a'_5	.360	0.14	.47	.02	.07	0.09
	6.02 in.^{2}		4.23 in.^{3}			6.89 in.^{4}

Thus the moment of inertia I_x of the area with respect to the XX axis is,

$$I_x = \Sigma I'_x = 6.89 \text{ in.}^4$$

Further, the total area is $a=\Sigma a'=6.02$ in.^{2} and the moment of the area with respect to the XX axis is $\Sigma(a'y_0)=4.23$ in.^{3} Hence, the distance, $\bar{y}$, of the centroid of the area from the XX axis is,

$$\bar{y}=\frac{\Sigma(a'y_0)}{a}=\frac{4.23}{6.02}=0.70 \text{ in.}$$

Therefore, the moment of inertia with respect to a line through the centroid and parallel to XX is given by the equation,

$$\begin{aligned}\bar{I}_x&=I_x-ad^2\\&=6.89-6.02\times(.70)^2\\&=6.89-2.95\\&=3.94 \text{ in.}^4\end{aligned}$$

PROBLEMS

270. A wooden column is built up of four 2-in. by 8-in. planks as shown in Fig. 300. Find the moment of inertia of the cross-section with respect to the centroidal axis XX. *Ans.* $I_x = 981$ in.^{4}

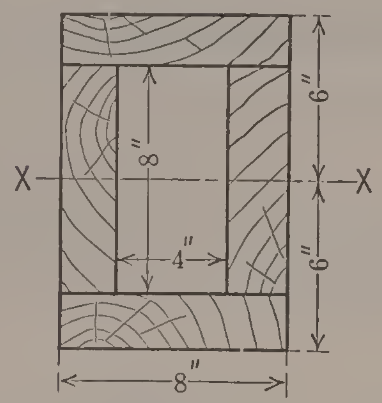

Fig. 300.

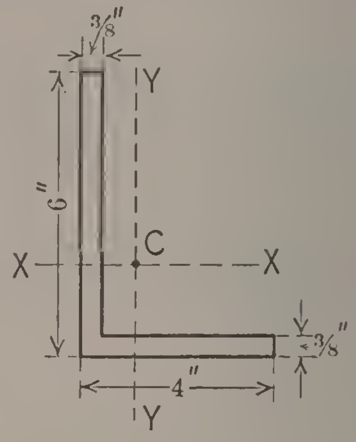

Fig. 301.

271. Find the moment of inertia of the angle section (Fig. 301) with respect to each of the centroidal axes parallel to the two legs of the angle.

272. In Fig. 302 is shown the cross-section of a standard 9-in. 21-lb. I-beam (fillets are neglected). Find the moments of inertia of the section with respect to the centroidal axes, XX and YY. *Ans.* $I_x = 84.9$ in.^{4}; $I_y = 5.16$ in.^{4}

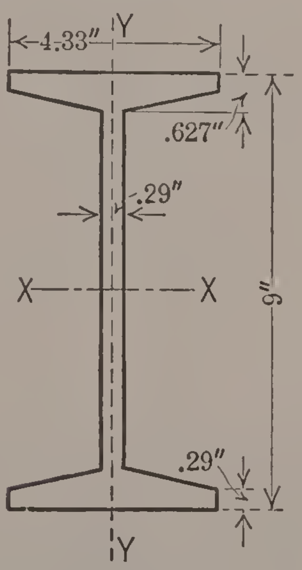

Fig. 302.

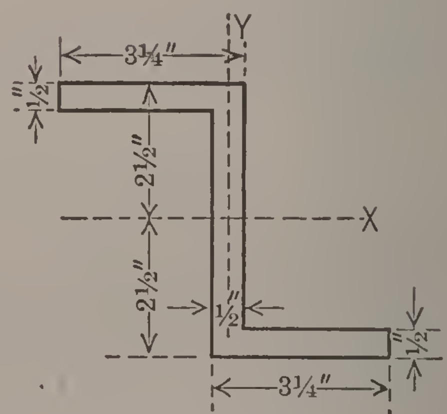

Fig. 303.

273. In Fig. 303 is shown the cross-section of a standard $3\frac{1}{4}$ in. by 5-in. Z-bar (fillets are neglected). Find the moments of inertia of the section with respect to the centroidal axes XX and YY.

274. In Fig. 304 is shown a built-up section made of a $\frac{1}{2}$-in. by 20-in. plate and four angles. Find the moment of inertia of the section with respect to the XX axis. *Ans.* $I_x = 1850$ in.^{4}

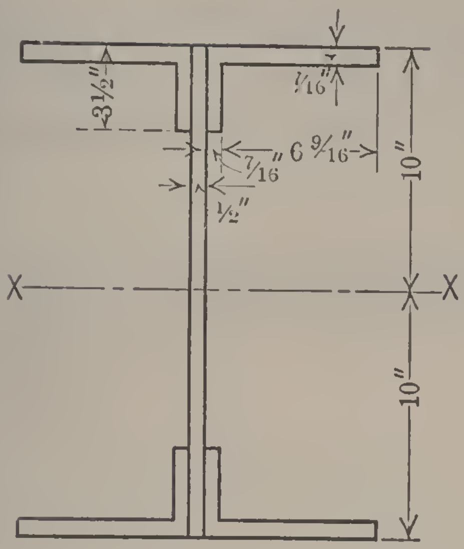

FIG. 304.

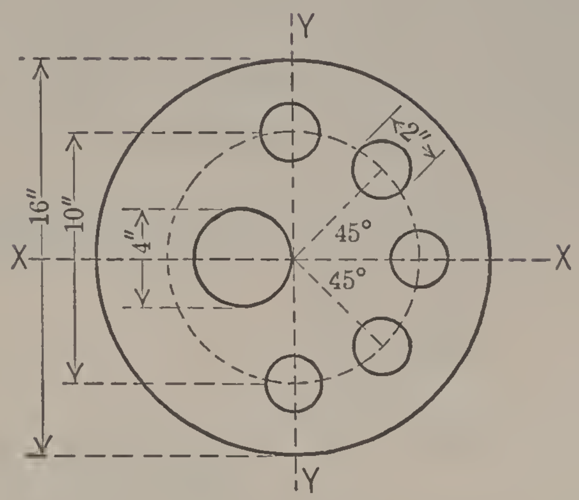

FIG. 305.

275. In Fig. 305 is represented a 16-in. circular plate in which there are drilled five 2-in. holes and one 4-in. hole as shown. Find the moment of inertia of the area of the holes with respect to the XX axis and also with respect to the YY axis. *Ans.* $I_x = 252$ in.^{4}

167. Approximate Method.—It is sometimes necessary to determine the moment of inertia of an area that has a boundary curve which cannot be defined by a mathematical equation. An approximate value of the moment of inertia of such an area may be found by the following method. For convenience, however, a simple area will be selected so that the approximate value of the moment of inertia as determined by this method may be compared with the exact value. Thus, let the moment of inertia of the area of a rectangle, with respect to an axis coinciding with its base, be found. The area may be divided into any convenient number of equal narrow strips parallel to the base, as shown in Fig. 306. (The narrower the strips the more closely will the result agree with the exact result.) Let the area be divided into ten such strips each 0.2 in. in width. The moment of inertia of the rectangle is equal to the sum of the moments of inertia of the strips. The moment of inertia of any particular strip with respect to the base of the rectangle is

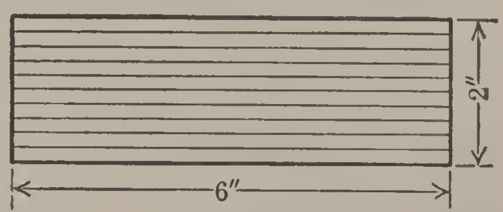

FIG. 306.

$$\tfrac{1}{12}\times 6\times(\tfrac{1}{5})^3+6\times\tfrac{1}{5}\times y^2,$$

where y is the distance of the centroid of the particular strip from the base. The first term is small and may be omitted without serious error. The moment of inertia of each strip then is approximately equal to the product of the area of the strip and the distance of its centroid from the base. Hence, the moment of inertia of the rectangle is,

$$I = \tfrac{6}{5}(.1^2+.3^2+.5^2+.7^2+.9^2+1.1^2+1.3^2+1.5^2+1.7^2+1.9^2)$$

$$= \tfrac{6}{5}\times 13.3$$

$$= 15.96 \text{ in.}^4.$$

According to Prob. 257 the exact value is,

$$I = \tfrac{1}{3}bh^3 = \tfrac{1}{3}\times 6\times 2^3 = 16 \text{ in.}^4$$

APPENDIX III

LOCALIZED STRESS; ITS OCCURRENCE, SIGNIFICANCE AND MEASUREMENT

By H. F. Moore[1] and F. B. Seely

168. Limitations of the Ordinary Methods of Analysis in Mechanics of Materials.—In Art. 126 and 143 it was pointed out that the use of the analyses and formulas given in the mechanics of materials as developed in Part I does not make possible the computation of *all* stresses existing in a machine part or a structural member. However, the ordinary formulas of the mechanics of materials are not to be thought of as invalid, but as presenting an incomplete statement or picture of the stress conditions in a member. For example, in an I-beam subjected to bending, neither the flexure formula $\left(s=\frac{Mc}{I}, \text{Art. } 34\right)$, which gives the tensile and compressive stresses, nor the formula for shearing stress $\left(s_s=\frac{V}{It}\int zda, \text{Art. } 40\right)$ take account of the intensity of bearing stress at the edge of a bearing block; in a shaft subjected to torsion the torsion formula $\left(s_s=\frac{Tc}{J}, \text{Art. } 26\right)$ cannot be used to determine the stress at the root of a keyway; in a bolt subjected to an axial tensile load, the axial load formula $\left(s=\frac{P}{a}, \text{Art. } 3\right)$ does not give a measure of the concentration of stress at the root of the thread, even when a is taken as the area at the root of the thread, etc.

169. Definition and Illustrations of Localized Stress.—Localized stress may be defined as stress existing over a small area of a machine part or a structural member, which stress is not deter-

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mined by the ordinary formulas of mechanics of materials as developed in Part I. Localized stresses occur at (1) the fillet of a shoulder in a shaft, (2) at the edge of a rivet hole, (3) at the root of the thread of a bolt, (4) at the edge of a non-metallic inclusion in a piece of steel, (5) at the edge of a bearing block, (6) at the bearing area between a car wheel and the rail, (7) at the root of a deep tool mark in a shaft, (8) at the sharp corner at the junction of head and shank of a bolt, etc.

170. The Significance of Localized Stress.—(*a*) *In Members Subjected to Static Loading.*—As was discussed in Art. 126 and 143, for machine parts and structural members subjected to steady load, or to load repeated a few times, localized stresses are, in general, of very little significance, unless the members are made of brittle material; ductile material yields at the points of high localized stress and hence the stress is transferred from the overstressed fibers to adjacent understressed fibers, and the deformation of the member as a whole is not appreciably affected. Therefore, when members made of ductile material are subjected to static loads the ordinary formulas of mechanics of materials may be regarded as giving a complete picture of the *significant* stresses in the more simple structural and machine members, although the secondary and localized stresses may be of much importance in the more complex structures, particularly those of large size and of new type.

(*b*) *In Members Subjected to Impact Loading.*—In machine parts or structural members subjected to impact or energy loading, localized stress is of more importance than it is in members subjected to steady loads. As discussed in Chapter XIII, the energy absorbed by a material when it is stressed is proportional to the *square* of the stress. This means that the small portions of a member where the localized stresses occur absorb an excessive amount of the energy load before the main portion of the member can be appreciably stressed and hence before the main portion can be made to absorb an appreciable share of the energy load. As a result, the small portion where a localized stress occurs is likely to be stressed above the yield-point of the material; this in turn still further localizes the absorption of energy, and there is danger of rupture even if the material is relatively ductile. The stresses developed by energy loads are discussed in Chapter XIII.

(c) *In Members Subjected to Repeated Loading.*—As discussed in Chapter XIV in machine parts and structural members that are subjected to repeated loading, localized stresses must frequently be considered to be the *significant* stresses, and hence the ordinary formulas in mechanics of material give only a very rough estimate of the values of the stresses that may cause damage to the material.

The above discussions indicate the need for methods of determining the magnitudes of localized stresses; some of these methods are discussed below.

171. Methods of Determining Magnitude of Localized Stress.—(a) *Mathematical Analysis.*—The simple mathematical analyses given in Part I serve to determine the average stresses over small areas of many common structural members and machine parts, but they do not determine the maximum stress *at a point.* When by the use of mathematical analyses it is attempted to get a more complete picture of stress distribution than is given by the ordinary analysis of mechanics of materials, especially when it is attempted to determine stresses or strains at sudden changes of outline of a part,—at fillets, shoulders, threads, and holes,—then the differential equations involved in the mathematical analysis become very difficult, if not impossible, of solution.

As an illustration of the determination of localized stress by means of mathematical analysis the reader is referred to the study by Inglis of the stresses at the edge of cracks in a plate.[1] In this paper mathematical analysis is applied to the determination of stresses and strains at the edges of elliptical holes in a plate, and a crack is regarded as an elliptical hole with a very short minor axis.

At the present development of our knowledge of mathematics, mathematical analysis can not be regarded as an available tool for the complete determination of localized stress in many machine parts and structural members.

(b) *Tests to Destruction of Models Made of Brittle Material.*—A method of stress analysis, which has been used with good success to give approximate values of the maximum unit-stress in complex shapes, consists in loading a model of the member made of a brittle material that has a flat (nearly straight) stress-strain

[1] Inglis, C. E. "Stresses in a plate due to the Presence of Cracks and Sharp Corners," Trans. Inst. of Naval Architects (British), Vol. LV, Pt. I, p. 219, 1913.

diagram up to the ultimate (such a material as hard cast iron, plaster of paris, or glass); from the same material is also made a model of simple shape, most commonly a tension test specimen. The specimen of simple shape and the model of complex shape are then tested to destruction; the test of the simple shape gives the ultimate strength of the material, and the test of the model gives with a fair degree of accuracy the load which produces this ultimate stress in the most stressed fibers of the model, and from these results the relation between load and maximum unit stress can be computed. The use of this method assumes that the deviation of the stress-strain diagram of the material from a straight line may be neglected.

An example of this method of stress-determination is found in the determination, first by Bach [2] and later by Kommers [3] of the shearing stresses in torsional members having non-circular cross-sections. Both Bach and Kommers used cast-iron specimens.

This method must be regarded as yielding only approximate results. The deviation from Hooke's law is considerable for all known materials when stressed up to rupture, and the effect of lateral restraint at the points of maximum (localized) stress causes the apparent values of the maximum unit-stress found in the test to be less than the true values. Moreover, by this method only the *maximum* unit-stress can be determined; this maximum unit-stress is, however, usually the most important stress to be determined. This method is simple, the model specimens are not very expensive to prepare, and valuable results may be secured by its use.

(*c*) *Yield-point Tests of Models Made of Ductile Material.*—The maximum localized stress in a member may be determined by a method somewhat similar to (*b*), in which a model of the piece made of ductile material with a well-marked yield-point is employed, such as ordinary low-carbon steel. The yield-point of the material is determined by the test of a tension specimen, a compression specimen, or a hollow torsion specimen depending on the kind of stress to which the model is subjected, and then load is applied to the model of the desired shape made of the same material. As load is applied the surface of the model is carefully watched for evidence

[2] Bach, C. "Elastizität und Festigkeit," Eighth edition, pp. 342 to 401.

[3] Kommers, J. B. "Torsion Tests of Cast Iron," American Machinist, May 28, 1914, p. 941.

of yielding. If the model is flat and if a surface can be left with the mill scale on it, the flaking off of this mill scale gives a fairly accurate indication of a localized stress equal to the yield-point of the material, and from the known value of the yield-point of the material and of the load on the model when flaking first occurs, the relation between load and unit-stress can be determined. If the surface of the model is painted with a wash of white portland cement the yielding can be detected by the cracking of the cement.

Instead of using the flaking of mill scale or of cement paint, as an indicator of the yield-point stress, the surface of the model may be polished, and the localized yield-point stress detected by a slight wrinkling of the polished surface, giving rise to what are known as "Luders' lines."

The yield-point method[4] of determining localized stress, like the brittle material method (*b*), determines only the maximum unit stress. It is usually desirable to use rather large models, and this necessitates the use of a large testing machine to develop in them the yield-point stress. This method, like method (*b*), should be regarded as yielding only approximate results, but it is very useful when more precise methods are not feasible.

(*d*) *Tests of Models Made of Plastic Material.*—Certain soft metals, such as soft copper, lead and type metal, acquire a permanent set when subjected to a very small stress, and the permanent set is approximately proportional to the stress applied. An approximate determination of the localized stress in irregular-shaped members can be made by making models of the irregular members out of soft copper, type metal or other plastic metal, laying off on the surface of the model a series of reference lines at known distances from each other, applying load, and, after removing the load, measuring the distortion between reference lines at various parts of the member. This measurement can sometimes be best made with the aid of a low-power microscope.

The relative distortion between gage lines at various parts of the model gives an approximate measure of the proportionate unit-strain developed at that location. Shearing unit-strain is measured by angular distortion between reference lines, tensile unit-strain or compressive unit-strain is measured by change of

[4] This method was used to determine the maximum stress in eye-bars by Epstein and Schwartz; the results of their tests were presented in 1911 as a student's thesis at the University of Illinois.

distance between lines. If the strain-distribution is known the stress-distribution can also be determined since the relation of stress to strain is known. At boundaries of the model this determination of stress from strain is usually quite simply accomplished.

In a distorted plastic model the distortions to be measurable must be of appreciable magnitude, and the distortions will themselves somewhat modify the distribution of stress throughout the model. This method of stress-analysis gives some idea of the general distribution of stress over the entire surface of the member, as well as the location and approximate determination of the maximum unit-stress. In using this method it is desirable to use as large models as is feasible, so that the distortion becomes measurable over a short gage length between reference lines.

The use of the distortion of plastic models in determining stress-distribution is illustrated in Bach's treatment of non-circular members subjected to torsion.[5]

(*e*) *Tests with Rubber Models.*—A method somewhat similar to (*d*) employs a rubber model similar in shape to the member to be studied. On the surface of the model reference lines are laid off, and the distortion between reference lines is measured while the member is resisting the load. This method is subject to the same inaccuracies as is method (*d*), and in addition is subject to error due to the fact that the stress-strain diagram for rubber deviates appreciably from a straight line, even for low stresses. The method is, however, very well adapted to show in a striking manner the general scheme of stress-distribution over the surface of a member. In using this method it is desirable to use as large models as is feasible, and, except for very simple shapes, such models are expensive.

Examples of the use of rubber models in determining stress-distribution are furnished by the work of Chiles and Kelley[6] on eyebars and plates with holes, and of Trelease[7] on stresses in flat slab structures.

[5] Bach, C. Elastizität und Festigkeit, Eighth edition, pp. 342–350.

[6] Chiles and Kelley, "The Resistance of Materials, The Effect of Sudden or Abrupt Changes in the Section on the Distribution of the Unit Stresses," Railway Mechanical Engineer, March, April, May, 1919.

[7] Trelease, "The Design of Concrete Flat Slabs," Proc. American Concrete Institute, Vol. VIII, 1912, p. 218.

(*f*) *The Use of the Strain Gage on Actual Structures or on Models.*—In any metal member the elastic strain in a gage length of 2 inches or more can be measured with a good degree of accuracy by means of a strain gage.[8] The strain gage is a special form of micrometer for measuring changes of length along a gage line on the surface of a member or specimen. The use of the strain gage to determine localized stress is especially applicable to full-size structural members, and large parts of machines. The instrument has been used successfully in determining strains in reinforced concrete structures, steel bridge members, car bolsters, built-up girders, and steel columns.

A limitation of the strain-gage method is the relatively long gage length which must be used. The ordinary strain gage cannot be used to determine localized stress at fillets, at the root of threads or at grooves in a shaft. The train-gage method is especially promising in the field of structural engineering.

An illustration of the use of the strain gage in determining localized stress is to be found in the work of Moore and Wilson on stresses in the webs of I-beams and girders.[9]

By the use of special very delicate extensometers attached to a specimen it is possible to measure strains over much shorter gage lengths than 2 inches. The investigations by Preuss of stress distribution round notches and holes in flat bars furnish a striking illustration of this method of stress analysis.[10]

(*g*) *The Use of Transparent Models, and Polarized Light.*—An elegant and accurate method for the determination of stress distribution in flat members subjected to stress in one plane employs model specimens made of glass, celluloid, or other transparent material, which are viewed by polarized light. Space forbids any detailed discussion of the optical problems involved in

[8] For a description of the strain gage and a discussion of the technique of its use, see Slater and Moore, "The Use of the Strain Gage in Testing Materials," Proc. Am. Soc. for Testing Materials, Vol. XIII, p. 1019 (1913).

[9] Moore and Wilson, "The Web Strength of I-beams and Girders," Bulletin 86, Engineering Experiment Station, University of Illinois.

[10] Preuss, E. "Versuche über die Spannungsverteilung in Gelochten Zugstaben," Zeit. Ver. Deut. Ing., 1912, p. 1780.

"Versuche über die Spannungsverteilung in Gekerbten Zugstaben," Zeit. Ver. Deut. Ing., 1913, p. 664. See also Johnson's Materials of Construction, fifth edition, p. 663.

the method.[11] In general, if a transparent substance is viewed by polarized light it appears to be of some definite color, and the color depends on the state of strain in the material. If the stress-distribution across a section is uniform the color will be uniform; if the stress distribution is variable there will be bands of various colors, merging into each other; if there are sudden changes of stress these bands are so close together that sharply marked dark spots show up. A glance at a stressed model specimen illuminated by polarized light shows whether there are points of high localized stress. A simple method of estimating the magnitude of localized stress consists in making from the same material as the model specimen a tension (or compression) test of a specimen of simple shape, in which the uniformity of color across the section shows a uniform stress-distribution. This auxiliary specimen is loaded until it shows some definite color red, say; it is then loaded further until the color changes through the spectrum to red again, and the difference in load for the change gives a measure of the stress required in the material to cause a "red-to-red" change. The model specimen of the shape to be studied is then tested, and the load necessary to cause a "red-to-red" change of color at any desired location is determined. This load then produces at the particular location a stress of known magnitude, and from the relation between the loads the localized stress at that location can be determined. Strictly speaking, it is not the stress which is determined directly, but the *difference* between principal strains at the location studied. However, at the boundaries, where the maximum localized stress frequently occurs, one principal strain is zero in which case the stress is found directly. In determining the maximum stress at a point not on the boundary it is necessary to measure the change in thickness at the location studied. This change in thickness is proportional to the *sum* of the principal strains at the location. Now having both the sum and the difference of the principal strains the maximum strain and the minimum strain can be determined.

[11] Coker, E. G. "Photo-elasticity," Engineering (London), Jan. 6, 1911, p. 1.

"The Optical Determination of Stress," Phil. Mag., Oct., 1910.

Heymans, Paul. "Photo-elasticity and its Applications to Engineering Problems." Publications of the Mass. Inst. of Tech., Serial No. 1, May, 1922.

"Stress Distribution in Rotating Gear Pinions as Determined by Photo-elastic Method," Jour. Mechanical Engineering, Mar., 1924, p. 129.

This is a very brief outline of only one of the methods used in the photo-elastic determination of stress,—one of the simplest and less accurate methods. More refined methods use mono-chromatic light for illuminating the specimen, and instead of causing a change of color in the specimen the load is applied until the desired location on the specimen is black, and under further load the location is again brought to " black " by a definite adjustment of the optical system, this adjustment being calibrated in terms of stress by a test on a simple tension or compression specimen.

The polarized-light method of stress-analysis can be made to yield results of a high degree of precision. The model specimens may be of small size, and comparatively inexpensive. In its present stage of development the method is limited to the study of flat members subjected to stresses in the plane of their flat surface;—the method could not be used, for example, to determine the localized stresses at the filleted shoulders of an axle.

(*h*) *Repeated Stress Tests.*—This method has been discussed in Art. 130 and 131.

(*i*) *Combination Methods of Stress-analysis.*—It is frequently possible to use two or more of the methods outlined in the foregoing paragraphs, as checks on each other; sometimes certain constants can be determined by an experimental method, after which mathematical analysis becomes possible. In determining the stress-distribution in the head of an eyebar the location of the maximum stress was determined by the yield-point test (method (*c*)), and with this location determined it was possible to apply mathematical analysis to determine the general stress-distribution.

A most striking illustration of the use of a combination of experiment and mathematical analysis is furnished by the work of Griffith and Taylor in studying the torsional stresses in airplane propeller blades.[12] The mathematical analysis of torsional stresses in members of irregular cross-section (such as airplane propeller blades) involves the use of differential equations that have not as yet been solved. However, it was observed that the differential equations for the deformations and stresses in a twisted bar of a given cross-section were, except for a constant term, the same as the differential equations for deflections and slopes at various

[12] Griffith and Taylor, Proc. Brit. Inst. of Mech. Engrs. 1917, Oct.-Dec., also, " Engineering " (London), Vol. 124, No. 3234, Dec. 21, 1917, p. 546.

points in an elastic film stretched over a hole of the same shape as the cross-section of the twisted bar, and deflected by a uniformly distributed normal pressure. The investigators used a soap film as an elastic film, stretched it over a brass plate, which had in it an opening of the shape of the cross-section of the airplane propeller blade, deflected the film by exhausting the air on one side of the plate, measured the deflection by means of a micrometer fitted with a soaped needle point, measured the slopes of the deflected film by means of a ray of light reflected from the film itself, and were thus able to determine the necessary constants for the computation of the shearing stresses in the airplane propeller blade when subjected to a given twisting moment. Their paper is recommended for careful study as a brilliant example of the combination of methods leading to a highly accurate stress analysis.

In conclusion, it is to be noted that this appendix is scarcely more than a list of possible methods of stress-analysis, and that further study, especially of the references given, will be necessary before the suggestions given here can become useful in actual determinations of localized stress.